EXISTENTIAL CLOSEDNESS OF $\overline{\mathbb{Q}}$ AS A GLOBALLY VALUED FIELD VIA ARAKELOV GEOMETRY

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Abstract. We prove that if integral polynomial equations together with open conditions on heights of solutions can be solved in a characteristic zero field with an abstract notion of height satisfying some natural axioms, then they can be solved in algebraic numbers.

In the language of Arakelov geometry, we characterise the weak*-closure of the set of functionals on the space of arithmetic line bundles on an arithmetic variety, that come from heights with respect to a closed point in the generic fiber. We give some applications using convex geometry.

Contents

1. Introduction

In this paper we study polynomial equations with height conditions. For a number $q \in \mathbb{Q}^\times$ its height is $\mathrm{ht}(q) := \log(\max(|a|, |b|))$, if $q = a/b$ for some coprime integers a, b. It measures the complexity of a number. One can extend the definition of height to algebraic numbers as follows. If $\alpha \in \overline{\mathbb{Q}}$ and $p(x) \in \mathbb{Z}[x]$ is its minimal polynomial with coprime coefficients decomposing as $p(x) = a(x-\alpha_1) \cdot \ldots \cdot (x-\alpha_n)$

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in $\mathbb{C}$, then the Mahler measure of p is

$$M(p) := |a| \prod_{i=1}^{n} \max\{1, |\alpha_i|\},$$

and the height of α is $\mathrm{ht}(\alpha) := \frac{1}{n} \log M(p)$. More generally, there is a natural notion of complexity, called the logarithmic Weil height $h_{\mathrm{Weil}}(\alpha_0 : \cdots : \alpha_n)$, of any element of $\mathbb{P}^n(\overline{\mathbb{Q}})$. Consider the following.

Question. When can we find algebraic numbers $\alpha_1, \ldots, \alpha_n$ satisfying some polynomial equations $f(\alpha_1, \ldots, \alpha_n) = 0$ together with height conditions of the form $\mathrm{ht}(g(\alpha_1, \ldots, \alpha_n)) = r$, for a polynomial $g(x_1, \ldots, x_n)$ and a real number r?

Without height conditions the answer is provided by Hilbert's Nullstellensatz: integral polynomial equations admit a solution in algebraic numbers if and only if the ideal generated by them does not contain the unit. In particular, there is an algebraic solution if and only if there is a solution in some algebraically closed field of characteristic zero. With conditions on heights the problem is much harder, as it is related to Lehmer's conjecture, heights of abelian varieties, Siegel's lemma, etc.

As there are only countably many algebraic numbers, not every real number can be presented as a height of one. Thus, we relax the above question, in that we consider open conditions of the form $a < \mathrm{ht}(g(\alpha_1, \ldots, \alpha_n)) < b$ instead. We give the following answer.

Theorem A. The field of algebraic numbers $\overline{\mathbb{Q}}$ is existentially closed as a globally valued field. In particular, for any problem:

$$\begin{cases} f_i(x_1, \ldots, x_n) = 0 \text{ for } i = 1, \ldots, k; \\ a_j < \mathrm{ht}(g_j(x_1, \ldots, x_n)) < b_j \text{ for } j = 1, \ldots, l; \end{cases}$$

where f_i, g_j are integral polynomials and a_j, b_j are real numbers, there is a solution in $\overline{\mathbb{Q}}$, if and only if there is a solution in a globally valued field with $h(2:1) = \log 2$.

Let us explain what this statement means. A globally valued field is a field K together with functions $h : \mathbb{P}^n(K) \to \mathbb{R}_{\geq 0}$ (for every natural n) satisfying a list of axioms specified in Definition 3.3.1. These axioms are satisfied by the the Weil logarithmic height, making $\overline{\mathbb{Q}}$ a globally valued field. Thus, the theorem can be interpreted as follows: the only obstruction for having a solution to a 'height problem', is when a contradiction can be deduced from the axioms of globally valued fields. Equivalently, it means that if X is a variety over a number field and the function field of X is equipped with a globally valued field structure h satisfying $h(2:1) = \log 2$, then there is a generic sequence of points $x_n \in X(\overline{\mathbb{Q}})$ such that for any tuple of non-zero rational functions $F_0, \ldots, F_m$ on X, we have

$$\lim_n h_{\mathrm{Weil}}(F_0(x_n) : \cdots : F_m(x_n)) = h(F_0 : \cdots : F_m).$$

Globally valued fields were defined by Ben Yaacov and Hrushovski around 2015 as a part of a far-reaching program of understanding model theory of global fields. These structures can be understood as models of some theory GVF_e formulated in unbounded continuous logic [4]. The original definition was different [6, 7] than the one we present here, however equivalent by [5] where we lay foundations of the theory. Theorem A gives an affirmative answer to a question raised by Ben Yaacov and Hrushovski, and is an arithmetic analogue of the corresponding function field result [6].

The main machinery behind our proof of Theorem A is Arakelov geometry. Initiated in [1] and developed in [17] and for higher dimensions in [18, 19], it provides a robust formalism for dealing with heights on varieties over number fields. Let $\mathscr{X}$ be a generically smooth *arithmetic variety*, i.e., a projective, integral scheme over the integers. Given a line bundle $\mathscr{L}$ on $\mathscr{X}$ together with a Hermitian metric h on the ambient complex analytic line bundle (satisfying some conditions), one can associate a function $h_{(\mathscr{L},h)} : \mathscr{X}(\overline{\mathbb{Q}}) \to \mathbb{R}$, that measures complexity of points of $\mathscr{X}$ with respect to the Hermitian line bundle $\overline{\mathscr{L}} := (\mathscr{L}, h)$. Weil logarithmic height can be presented as a uniform limit of functions of this form. Isomorphism classes of Hermitian line bundles on $\mathscr{X}$ form an ordered abelian group $\widehat{\mathrm{Pic}}(\mathscr{X})$, where $\overline{\mathscr{L}} \geq 0$ if and only if $\mathscr{L}$ has a global section with norm smaller or equal one everywhere. In particular, each point $x \in \mathscr{X}(\overline{\mathbb{Q}})$ defines a function $h_{(-)}(x) : \widehat{\mathrm{Pic}}(\mathscr{X}) \to \mathbb{R}$ sending $\overline{\mathscr{L}}$ to $h_{\overline{\mathscr{L}}}(x)$. We denote by $\widehat{\mathrm{Pic}}(\mathscr{X})^\vee_{+,1}$ the set of linear maps $\widehat{\mathrm{Pic}}(\mathscr{X}) \to \mathbb{R}$ that take the positive cone to non-negative numbers and that take the trivial line bundle with the trivial metric to 1. Using various tools from Arakelov geometry [34, 31, 14, 12, 30, 27], we prove the following.

Theorem B. Let $\mathscr{U} \subset \mathscr{X}$ be a non-empty Zariski open set. Functionals of the form $h_{(-)}(x)$ for $x \in \mathscr{U}(\overline{\mathbb{Q}})$ are weak*-dense in $\widehat{\mathrm{Pic}}(\mathscr{X})^\vee_{+,1}$.

To deduce Theorem A from Theorem B, we relate globally valued field structures on a finitely generated field F over $\mathbb{Q}$ to elements of $\widehat{\mathrm{Pic}}(\mathscr{X})^\vee_{+,1}$, for $\mathscr{X}$ having its function field isomorphic to F. For an arithmetic variety $\mathscr{X}$, we denote by $\mathrm{ADiv}_{\mathbb{Q}}(\mathscr{X})$ the ordered group of arithmetic $\mathbb{Q}$-divisors with continuous Green's functions. Each Hermitian line bundle $(\mathscr{L}, h)$ with a rational section s determines an arithmetic divisor $(\mathrm{div}(s), -\log |s|^2_h)$, but arithmetic $\mathbb{Q}$-divisors with continuous Green's functions are slightly more general. We prove that the colimit $\mathrm{ADiv}_{\mathbb{Q}}(F)$ of ordered groups $\mathrm{ADiv}_{\mathbb{Q}}(\mathscr{X})$ over the system of all arithmetic varieties birational to $\mathscr{X}$ is a lattice (as an order). This is related to the formalism of (generalised or b-) Weil functions from [29, 28]. Using results from [5] this allows us to conclude with the following.

Theorem C. Let F be a finitely generated field over $\mathbb{Q}$. There is a bijection between the set of globally valued field structures on F, and the set of positive normalised functionals $\mathrm{ADiv}_{\mathbb{Q}}(F) \to \mathbb{R}$ that vanish on principal divisors.

Thus, globally valued field structures on F are also in bijection with the limit of $\widehat{\mathrm{Pic}}(\mathscr{X})^\vee_{+,1}$ over the system of arithmetic varieties equipped with an isomorphism of their function field with F. Using the above theorems and elementary convex geometry, one immediately gets some consequences for values of heights on algebraic numbers. For example, it follows that for any $\overline{\mathscr{L}}$ on an arithmetic variety $\mathscr{X}$ with a non-empty open subset $\mathscr{U}$, if $\mathscr{L} \otimes \mathbb{Q}$ is big, then the set $\{h_{\overline{\mathscr{L}}}(x) : x \in \mathscr{U}\}$ is dense in the interval bounded below by the essential infimum of $\overline{\mathscr{L}}$. Also, for finitely many Hermitian line bundles $\overline{\mathscr{L}}_1, \dots, \overline{\mathscr{L}}_n$ on $\mathscr{X}$ (assuming one of them is big) it easily follows that the set of limit points of tuples $(h_{\overline{\mathscr{L}}_1}(x_i), \dots, h_{\overline{\mathscr{L}}_n}(x_i)) \in \mathbb{R}^n$, for generic sequences $(x_i)_{i \in \mathbb{N}}$, is convex. See [6] for some consequences for the equidistribution of Galois orbits. In Section 3.4 we give an interpretation of the essential infimum function in terms of convex geometry, and we prove that differentiability of the essential infimum is equivalent to logarithmic equidistribution, see Theorem 3.4.8 and Corollary 3.4.12.

There are many approaches to Arakelov geometry in the literature including [11, 25, 26, 33]. We choose to work with arithmetic $\mathbb{R}$-divisors of C^0-type from [25] in this paper, because in that setup it is convenient to pass between Hermitian line bundles and real combinations of arithmetic divisors. Note that all the results extend easily to the adelic framework from [26] by Theorem 4.1.3 in loc. cit. Furthermore, since adelic line bundles from [33] are defined as limits with respect to the boundary topology (see [33, Section 1.1.3]), GVF functionals automatically extend to those as well. Let us mention that a different approach to global geometry, closely related to globally valued fields by results of [5], was developed by Gubler [20], and by Chen and Moriwaki [15, 16].

Existential closedness of $\overline{\mathbb{Q}}$ sets the goal of axiomatizing existentially closed models of the theory GVF_e. The model companion is conjectured to exist [5], and it is possible that it is tame in some model theoretic sense. For example, by [7], GVF structures are quantifier-free stable (i.e., quantifier-free formulas do not have the order property in the continuous logic sense). Stability (and simplicity) of various structures had some striking applications in the past [21, 22, 24, 10] and it would be interesting to see new applications in the realm of heights. Moreover, existence of the model companion would imply decidability of problems appearing in the statement of Theorem A (provided that the heights of all variables are bounded).

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2. Preliminaries

2.1. **Arithmetic line bundles and divisors.** Let $\mathscr{X}$ be a generically smooth *arithmetic variety* (integral scheme that is projective over $\mathrm{Spec}(\mathbb{Z})$) of dimension $d+1$ and let $X = \mathscr{X}_{\mathbb{Q}}$ be the generic fiber. In this section we start with some preliminaries on Hermitian line bundles on arithmetic varieties and arithmetic $\mathbb{R}$-divisors. For more thorough treatment of these topics see [11, 12, 23, 25, 30].

By $\mathscr{X}(\mathbb{C})$ we mean the complex analytification of the base change $\mathscr{X}_{\mathbb{C}} = \mathscr{X} \otimes \mathbb{C}$. It is in a natural bijection with the set of maps $\mathrm{Spec}(\mathbb{C}) \to \mathscr{X}$. Complex conjugation induces a map $\mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(\mathbb{C})$ and by precomposition it induces *complex conjugation* $F_\infty : \mathscr{X}(\mathbb{C}) \to \mathscr{X}(\mathbb{C})$. If $p \in \mathscr{X}$ is a point in a scheme, we write $\kappa(p)$ for the residue field. The function field of an integral scheme $\mathscr{X}$ is denoted by $\kappa(\mathscr{X})$. If $\sigma : \kappa(x) \to \mathbb{C}$ is an embedding of fields, we denote by x^σ the point of $\mathscr{X}(\mathbb{C})$ corresponding to the composition

$$\mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(\kappa(x)) \to X \to \mathscr{X}.$$

Definition 2.1.1. A *Hermitian line bundle* $\overline{\mathscr{L}} = (\mathscr{L}, |\cdot|_{\overline{\mathscr{L}}})$ on $\mathscr{X}$ is a pair consisting of a line bundle $\mathscr{L}$ on $\mathscr{X}$ and a smooth Hermitian metric of real type $|\cdot|_{\overline{\mathscr{L}}}$

on the complex analytification of the line bundle $\mathscr{L}_{\mathbb{C}}$ on $\mathscr{X}_{\mathbb{C}}$. Recall that h is of *real type*, if for all $x \in M$ with $\overline{x} := F_\infty(x)$ and for all sections $s, s' \in L_x$ we have

$$h_x(s,s') = \overline{h_{\overline{x}}(F_\infty(s), F_\infty(s'))},$$

where the action on germs of sections is defined by precomposition with F_∞. A section $s \in H^0(\mathscr{X}, \mathscr{L})$ is *(strictly) small*, if $|s|_{\overline{\mathscr{L}}} \leq 1$ (resp. $|s|_{\overline{\mathscr{L}}} < 1$) on $\mathscr{X}(\mathbb{C})$.

Remark 2.1.2. For the rest of this section (unless stated otherwise) we impose **stronger** assumptions on $\mathscr{X}$. Namely here $\mathscr{X}$ is a generically smooth, **normal** arithmetic variety of dimension $d+1$. This is harmless as one can always pass to a normalisation of $\mathscr{X}$, but is convenient, as Cartier divisors have unique associated Weil divisors.

If $\mathbb{K}$ is one of $\mathbb{Q}, \mathbb{R}$, then a $\mathbb{K}$-*Cartier divisor* on $\mathscr{X}$ is an element of the group $\mathrm{Div}(\mathscr{X})_{\mathbb{K}}$. A $\mathbb{K}$-*rational function* is an element of $\kappa(\mathscr{X})^{\times}_{\mathbb{K}}$. Let $\mathscr{D} = \sum_i \alpha_i \mathscr{D}_i \in \mathrm{Div}(\mathscr{X})_{\mathbb{K}}$ for $\alpha_i \in \mathbb{K}$ and Cartier divisors $\mathscr{D}_i$. If f_i is a local equation for $\mathscr{D}_i$ for all i, then we call $\prod_i f_i^{\alpha_i} \in \kappa(\mathscr{X})^{\times}_{\mathbb{K}}$ a local equation for $\mathscr{D}$. On the other hand, if $f \in \kappa(\mathscr{X})^{\times}_{\mathbb{K}}$, then we write $\mathrm{div}(f)$ for the corresponding $\mathbb{K}$-Cartier divisor. For $\mathscr{D} \in \mathrm{Div}(\mathscr{X})_{\mathbb{K}}$ we write $\mathscr{D}_{\mathbb{Q}} \in \mathrm{Div}(\mathscr{X}_{\mathbb{Q}})_{\mathbb{K}}$ for its restriction to the generic fiber.

Definition 2.1.3. Let $\mathscr{D}$ be an $\mathbb{R}$-Cartier divisor on $\mathscr{X}$. A *$\mathscr{D}$-Green's function on $\mathscr{X}$ of C^0-type* (respectively C^∞-type) is a function $g : \mathscr{U}(\mathbb{C}) \to \mathbb{R}$ for some Zariski open $\mathscr{U} \subset \mathscr{X}$ such that if $\mathscr{D} = \sum a_i \mathscr{D}_i$ for some real a_i's and Cartier divisors $\mathscr{D}_i$'s, then for every $p \in \mathscr{X}(\mathbb{C})$ there is a Zariski open $\mathscr{V} \subset \mathscr{X}$ with local equations f_i's for $\mathscr{D}_i$'s, such that $p \in \mathscr{V}(\mathbb{C}) \subset \mathscr{X}(\mathbb{C})$ and the function

$$g(x) + \sum_i a_i \log |f_i(x)|^2$$

restricted to the intersection of $\mathscr{U}(\mathbb{C})$ and $\mathscr{V}(\mathbb{C})$, extends to a continuous (respectively smooth) and conjugation-invariant function on $\mathscr{V}(\mathbb{C})$. We identify arithmetic $\mathbb{R}$-divisors of C^0-type $(\mathscr{D}, g) = (\mathscr{D}, g')$ if $g : \mathscr{U}(\mathbb{C}) \to \mathbb{R}, g' : \mathscr{U}'(\mathbb{C}) \to \mathbb{R}$ coincide on the intersection $(\mathscr{U} \cap \mathscr{U}')(\mathbb{C})$.

Definition 2.1.4. An *arithmetic $\mathbb{R}$-divisor of C^0-type (resp. C^∞-type) on $\mathscr{X}$* is a pair $\overline{\mathscr{D}} = (\mathscr{D}, g)$, where $\mathscr{D}$ is an $\mathbb{R}$-Cartier divisor on $\mathscr{X}$ and g is a $\mathscr{D}$-Green's function on $\mathscr{X}$ (of C^∞-type). An arithmetic $\mathbb{R}$-divisor of C^0-type is called *principal*, if it can be written as an $\mathbb{R}$-combination of arithmetic $\mathbb{R}$-divisors (of C^∞-type) $\widehat{\mathrm{div}}(f) := (\mathrm{div}(f), -\log|f|^2)$ for some rational functions f on $\mathscr{X}$. Also, $(\mathscr{D}, g)$ is an *arithmetic $\mathbb{Z}$-divisor [resp. $\mathbb{Q}$-divisor] of C^0-type (or C^∞-type)* if it is an arithmetic $\mathbb{R}$-divisor of the corresponding type, with $\mathscr{D}$ being a Cartier [resp. $\mathbb{Q}$-Cartier] divisor on $\mathscr{X}$.

Notation 2.1.5. Arithmetic $\mathbb{R}$-divisors of C^0-type form an abelian group (with component-wise addition) and this group is denoted by $\mathrm{ADiv}_{\mathbb{R}}(\mathscr{X})$. The subgroup of principal arithmetic $\mathbb{R}$-divisors of C^0-type is denoted by $\mathrm{RDiv}_{\mathbb{R}}(\mathscr{X})$. The quotient $\mathrm{ADiv}_{\mathbb{R}}(\mathscr{X}) / \mathrm{RDiv}_{\mathbb{R}}(\mathscr{X})$ is denoted by $\mathrm{APic}_{\mathbb{R}}(\mathscr{X})$.

We equip $\mathrm{ADiv}_{\mathbb{R}}(\mathscr{X})$ with the colimit topology coming from the system of bilinear maps $V \times C(\mathscr{X}(\mathbb{C})/F_\infty) \to \mathrm{ADiv}_{\mathbb{R}}(\mathscr{X})$, where V vary among finite dimensional subspaces of $\mathrm{ADiv}_{\mathbb{R}}(\mathscr{X})$ equipped with the Euclidean topology and $C(\mathscr{X}(\mathbb{C})/F_\infty)$ is the space of continuous real-valued functions on $\mathscr{X}(\mathbb{C})/F_\infty$ (equipped with the supremum norm topology) seen as arithmetic $\mathbb{R}$-divisors of C^0-type on $\mathscr{X}$ with the zero divisor.

Remark 2.1.6. Let $\phi : \mathscr{Y} \to \mathscr{X}$ be a map of generically smooth, normal arithmetic varieties and let $\overline{\mathscr{D}} = (\mathscr{D}, g)$ be an arithmetic $\mathbb{R}$-divisor of C^0-type on $\mathscr{X}$. If the pullback $\phi^*\mathscr{D}$ exists and g can be defined on an open subset $\mathscr{U}$ of $\mathscr{X}$ whose preimage is non-empty in $\mathscr{Y}$, one can equip $\phi^*\mathscr{D}$ with a Green's function being the pullback of g.

2.2. **Positivity of arithmetic divisors.**

Definition 2.2.1. Let F be a field.

(1) We call a function $v : F^\times \to \mathbb{R}$ a *non-Archimedean valuation* if $v(xy) = v(x)+v(y)$ and v satisfies the ultrametric inequality $v(x+y) \geq \min(v(x), x(y))$ (with the convention $v(0) = \infty$).
(2) By an *Archimedean valuation* on F we mean a function $v : F^\times \to \mathbb{R}$ of the form $v(x) = -\log|\sigma(x)|$, where $\sigma : F \to \mathbb{C}$ is an embedding of fields.
(3) By a *valuation* we mean either a non-Archimedean or an Archimedean valuation on a field.

If $\mathscr{X}$ is an arithmetic variety and v is a non-Archimedean valuation on $\kappa(\mathscr{X})$, we write $\mathrm{supp}(v) \in \mathscr{X}$ for the image of the maximal ideal of the valuation ring $\mathscr{O}_v \subset \kappa(\mathscr{X})$ via the unique map ϕ making the following diagram commute:

$$\begin{tikzcd} \mathrm{Spec}(\kappa(\mathscr{X})) \arrow[r] \arrow[d] & \mathscr{X} \arrow[d] \\ \mathrm{Spec}(\mathscr{O}_v) \arrow[r] \arrow[ur, dashed, "\phi"] & \mathrm{Spec}(\mathbb{Z}). \end{tikzcd}$$

Such a map always exists by the valuative criterion of properness.

Definition 2.2.2. Let $\mathscr{D}$ be an $\mathbb{R}$-Cartier divisor on $\mathscr{X}$ and let v be a non-Archimedean valuation on $\kappa(\mathscr{X})$. Let $p = \mathrm{supp}(v) \in \mathscr{X}$ and let $f \in \kappa(\mathscr{X})^\times_{\mathbb{R}}$ be a local equation for $\mathscr{D}$ at p. We then write $\beta(v, \mathscr{D})$ for the value $v(f)$ which is the $\mathbb{R}$-linear (multiplicative) extension of the valuation $v : \kappa(\mathscr{X})^\times \to \mathbb{R}$. Moreover, if $\overline{\mathscr{D}}$ is an arithmetic $\mathbb{R}$-divisor of C^0-type on $\mathscr{X}$ we define $\beta(v, \overline{\mathscr{D}}) := \beta(v, \mathscr{D})$. On the other hand, if v is an Archimedean valuation on $\kappa(\mathscr{X})$ induced by $\sigma : \kappa(\mathscr{X}) \to \mathbb{C}$, then we define $\beta(v, \overline{\mathscr{D}}) := \frac{1}{2}g(p)$, where p is the point of $\mathscr{X}(\mathbb{C})$ coming from the composition

$$\mathrm{Spec}(\mathbb{C}) \xrightarrow{\mathrm{Spec}(\sigma)} \mathrm{Spec}(\kappa(\mathscr{X})) \longrightarrow \mathscr{X}.$$

Note that g is defined on p since $\mathrm{Spec}(\kappa(\mathscr{X}))$ is the generic point of $\mathscr{X}$. Furthermore, since g is conjugation-invariant, the value $g(p)$ does not depend on the choice of σ inducing v.

In particular, if $\mathscr{D}$ is an $\mathbb{R}$-Cartier divisor on $\mathscr{X}$, then the associated $\mathbb{R}$-Weil divisor is of the form

$$\sum_\Gamma \mathrm{ord}_\Gamma(\mathscr{D})\Gamma,$$

where the sum is taken over all codimension one integral subschemes of $\mathscr{X}$ and $\mathrm{ord}_\Gamma(\mathscr{D}) = \beta(\mathrm{ord}_\Gamma, \mathscr{D})$. This makes sense, as we assume that $\mathscr{X}$ is normal, which implies that the order of vanishing at an integral subscheme Γ of codimension one is a valuation on $\kappa(\mathscr{X})$.

Lemma 2.2.3. Let $\mathscr{D}$ be an $\mathbb{R}$-Cartier divisor on $\mathscr{X}$. The following are equivalent.

(1) $\mathscr{D}$ is a positive $\mathbb{R}$-combination of effective Cartier divisors.
(2) For all $p \in \mathscr{X}$ if $f \in \kappa(\mathscr{X})^{\times}_{\mathbb{R}}$ is a local equation for $\mathscr{D}$ at p, then f is in the subsemigroup of $\kappa(\mathscr{X})^{\times}_{\mathbb{R}}$ generated by elements of the form g^r with $g \in \mathscr{O}_{\mathscr{X},p} \setminus \{0\}$ and $r > 0$.
(3) For all non-Archimedean valuations v on $\kappa(\mathscr{X})$ we have $\beta(v, \mathscr{D}) \geq 0$.
(4) If $\sum_{\Gamma} a_{\Gamma}\Gamma$ is the $\mathbb{R}$-Weil divisor associated to $\mathscr{D}$, then all a_{Γ} are non-negative.

Moreover, the same holds (with $\mathbb{Q}$-tensors instead of $\mathbb{R}$-tensors) for a $\mathbb{Q}$-Cartier divisor $\mathscr{D}$.

Proof. Implications (1) $\implies$ (2) $\implies$ (3) $\implies$ (4) are obvious. The implication (4) $\implies$ (1) is [15, Proposition 2.4.16]. Implications in the $\mathbb{Q}$ case are skipped. □

Definition 2.2.4. Let $\mathscr{D}$ be an $\mathbb{R}$-Cartier divisor on $\mathscr{X}$. We say that $\mathscr{D}$ is *effective*, if it satisfies the equivalent conditions from Lemma 2.2.3. In this case we write $\mathscr{D} \geq 0$. Similarly, if $\mathscr{E}$ is another $\mathbb{R}$-Cartier divisor on $\mathscr{X}$, we write $\mathscr{D} \geq \mathscr{E}$, if $\mathscr{D} - \mathscr{E} \geq 0$. We also define the space of rational functions having poles at most given by $\mathscr{D}$ as

$$H^0(\mathscr{X}, \mathscr{D}) := \{f \in \kappa(\mathscr{X})^{\times} : \mathscr{D} + \operatorname{div}(f) \geq 0\} \cup \{0\}.$$

A similar definition applies to $\mathbb{R}$-Cartier divisors on $X = \mathscr{X}_{\mathbb{Q}}$.

Definition 2.2.5. Let $\overline{\mathscr{D}} = (\mathscr{D}, g)$ be an arithmetic $\mathbb{Z}$-divisor of C^0-type on $\mathscr{X}$ with $\mathscr{D}$ effective and let $x \in X = \mathscr{X}_{\mathbb{Q}}$ be a closed point of the generic fiber of $\mathscr{X}$. Let $\mathscr{C} = \overline{\{x\}} \subset \mathscr{X}$ be the Zariski closure of x equipped with the reduced scheme structure. If $x \notin \operatorname{supp}(\mathscr{D})$ and $\mathscr{D}$ is a Cartier divisor on $\mathscr{X}$ the *height* of x with respect to $\overline{\mathscr{D}}$ is defined to be

$$h_{\overline{\mathscr{D}}}(x) := \frac{1}{[\kappa(x) : \mathbb{Q}]}\Big(\log \#\big(\mathscr{O}_{\mathscr{C}}(\mathscr{D})/\mathscr{O}_{\mathscr{C}}\big) + \frac{1}{2}\sum_{\sigma : \kappa(x) \to \mathbb{C}} g(x^{\sigma})\Big),$$

where we treat $\mathscr{O}_{\mathscr{C}}(\mathscr{D})/\mathscr{O}_{\mathscr{C}}$ as a module over the ring of functions on the zero-dimensional scheme $\mathscr{C} \cap \mathscr{D}$ (the closed subscheme of $\mathscr{C}$ cut out by the local equations for $\mathscr{D}$). For a general $\overline{\mathscr{D}}$, there is a decomposition $\overline{\mathscr{D}} = \sum_{i=1}^{m} \alpha_i \overline{\mathscr{D}}_i + \widehat{\operatorname{div}}(f)$ with α_i real, $\mathscr{D}_i$ effective Cartier, such that $x \notin \operatorname{supp}(\mathscr{D}_i)$ for all $i = 1, \ldots, m$ and where f is an $\mathbb{R}$-rational function on $\mathscr{X}$. Then we put

$$h_{\overline{\mathscr{D}}}(x) := \sum_{i=1}^{m} \alpha_i h_{\overline{\mathscr{D}}_i}(x).$$

Note that this is well defined, and only depends on the class of $\overline{\mathscr{D}}$ in $\operatorname{APic}_{\mathbb{R}}(\mathscr{X})$, by the product formula on number fields. For details, see [26, Section 4.2]. Also, we define the *arithmetic degree* of $\overline{\mathscr{D}}$ with respect to x as the quantity $\widehat{\deg}(\overline{\mathscr{D}}|\overline{\{x\}})$ satisfying

$$h_{\overline{\mathscr{D}}}(x) = \frac{\widehat{\deg}(\overline{\mathscr{D}}|\overline{\{x\}})}{\deg(x)}.$$

If $x \in \mathscr{X}(\overline{\mathbb{Q}})$ we write $h_{\overline{\mathscr{D}}}(x)$ for the height of the corresponding closed point of X. It is easy to check that for a fixed x the function $h_{\overline{\mathscr{D}}}(x)$ is continuous in $\overline{\mathscr{D}}$.

Lemma 2.2.6. Let $x \in X = \mathscr{X} \otimes \mathbb{Q}$ be a closed point and let $\overline{\mathscr{N}}$ be the zero divisor with the Green's function 2. Then

$$h_{\overline{\mathscr{N}}}(x) = 1.$$

Proof. Follows directly from the definition. □

Note that Green's function in $\overline{\mathscr{N}}$ is 2 instead of 1, because in Definition 2.1.3 we used logarithms of squares of local equations.

Lemma 2.2.7. Assume that $\mathscr{X}'$ is a generically smooth, normal arithmetic variety. Let $f : \mathscr{X}' \to \mathscr{X}$ be a generically finite map and let $x' \in X' = \mathscr{X}' \otimes \mathbb{Q}$ be a closed point. Then for any arithmetic $\mathbb{R}$-divisor of C^0-type $\overline{\mathscr{D}}$ on $\mathscr{X}$ we have

$$h_{f^*\overline{\mathscr{D}}}(x') = h_{\overline{\mathscr{D}}}(f(x')).$$

Proof. Follows from the projection formula [11, Proposition (2.9)] and linearity. □

Definition 2.2.8. We use the following notions regarding an arithmetic $\mathbb{R}$-divisor of C^0-type $\overline{\mathscr{D}} = (\mathscr{D}, g)$ on $\mathscr{X}$ with generic fiber $X = \mathscr{X}_{\mathbb{Q}}$.

(1) We write $\overline{\mathscr{D}} \geq 0$ and call $\overline{\mathscr{D}}$ *effective*, if $\mathscr{D} \geq 0$ and $g \geq 0$. Note that this does not depend on the choice of g since if g' extends g and $g'(x) < 0$ for some $x \in \mathscr{X}(\mathbb{C})$, then the set $\{y \in \mathscr{X}(\mathbb{C}) : g'(y) < 0\}$ is open, so it has to intersect the domain of g which is a dense open subset of $\mathscr{X}(\mathbb{C})$. If $\overline{\mathscr{E}}$ is another arithmetic $\mathbb{R}$-divisor of C^0-type on $\mathscr{X}$, then $\overline{\mathscr{D}} \geq \overline{\mathscr{E}}$ means that $\overline{\mathscr{D}} - \overline{\mathscr{E}} \geq 0$.

(2) There is a natural norm $|\cdot|^{\overline{\mathscr{D}}}$ associating to each rational function $f \in H^0(\mathscr{X}, \mathscr{D})$ a function

$$|f|^{\overline{\mathscr{D}}} : \mathscr{X}(\mathbb{C}) \to \mathbb{R},$$

which is the unique continuous extension of $|f| \cdot \exp(-g/2)$ on $\mathscr{X}(\mathbb{C})$. For well definiteness, see [26, Proposition 1.4.2]. It induces a supremum norm

$$\|\cdot\|_{\sup}^{\overline{\mathscr{D}}} : H^0(\mathscr{X}, \mathscr{D}) \to \mathbb{R}.$$

A rational function $f \in H^0(\mathscr{X}, \mathscr{D})$ is called *small*, if $\|f\|_{\sup}^{\overline{\mathscr{D}}} \leq 1$. It is called *strictly small* if $\|f\|_{\sup}^{\overline{\mathscr{D}}} < 1$. The set of small sections is denoted by $\widehat{H}^0(\mathscr{X}, \overline{\mathscr{D}})$.

(3) Assume that $\overline{\mathscr{D}}$ is an arithmetic $\mathbb{Z}$-divisor of C^∞-type. Let L be the (complex analytic) line bundle over $\mathscr{X}(\mathbb{C})$ induced by the base change of $\mathscr{O}_X(\mathscr{D}_{\mathbb{Q}})$ to $\mathbb{C}$. Then for a Zariski open $\mathscr{U} \subset \mathscr{X}$ where $d \in \kappa(\mathscr{X})^\times$ is a local equation for $\mathscr{D}$ and a section s of L over $\mathscr{U}(\mathbb{C})$ one can define a norm of s by presenting $s = f \cdot d^{-1}$ for some holomorphic function $f : \mathscr{U}(\mathbb{C}) \to \mathbb{C}$ and by putting

$$|s|_{\overline{L}} := |f| \cdot \exp(-(g + \log|d|^2)/2),$$

where by $g + \log|d|^2$ we mean the unique continuous (even smooth) extension of this function to $\mathscr{U}(\mathbb{C})$. In fact, this metric comes from a unique Hermitian product on L and the Chern form of the pair $(L, |\cdot|_{\overline{L}})$ is denoted by $c_1(\overline{\mathscr{D}})$. The corresponding Hermitian line bundle is denoted by $\mathscr{O}(\overline{\mathscr{D}})$.

(4) The *volume* of an $\mathbb{R}$-divisor $\mathscr{D}_{\mathbb{Q}}$ on the generic fiber X is

$$\mathrm{vol}(\mathscr{D}_{\mathbb{Q}}) := \limsup_{m \to \infty} \frac{\dim_{\mathbb{Q}} H^0(X, m\mathscr{D}_{\mathbb{Q}})}{m^d/d!}.$$

(5) The *arithmetic volume* of $\overline{\mathscr{D}}$ is defined to be the following number

$$\widehat{\mathrm{vol}}(\overline{\mathscr{D}}) := \limsup_{m\to\infty} \frac{\log \#\widehat{H}^0(\mathscr{X}, m\overline{\mathscr{D}})}{m^{d+1}/(d+1)!}.$$

The arithmetic volume depends only on the class of $\overline{\mathscr{D}}$ in $\mathrm{APic}_{\mathbb{R}}(\mathscr{X})$.

Lemma 2.2.9. An arithmetic $\mathbb{R}$-divisor of C^0-type $\overline{\mathscr{D}}$ on $\mathscr{X}$ is effective if and only if for every valuation v on $\kappa(\mathscr{X})$ (non-Archimedean or Archimedean) we have $\beta(v, \overline{\mathscr{D}}) \geq 0$.

Proof. This is by (3) in Lemma 2.2.3. □

Definition 2.2.10. We collect here the properties of arithmetic $\mathbb{R}$-divisors of C^0-type on $\mathscr{X}$ crucial for the proof of Theorem 3.1.4.

(1) An arithmetic $\mathbb{R}$-divisor of C^0-type $\overline{\mathscr{D}}$ on $\mathscr{X}$ is *big*, if $\widehat{\mathrm{vol}}(\overline{\mathscr{D}}) > 0$.
(2) An arithmetic $\mathbb{R}$-divisor of C^0-type $\overline{\mathscr{E}}$ on $\mathscr{X}$ is *pseudo-effective*, if for all big arithmetic $\mathbb{R}$-divisors of C^0-type $\overline{\mathscr{B}}$ on $\mathscr{X}$, the sum $\overline{\mathscr{B}} + \overline{\mathscr{E}}$ is big.
(3) We say that $\overline{\mathscr{D}} = (\mathscr{D}, g) \in \mathrm{ADiv}_{\mathbb{R}}(\mathscr{X})$ is *nef*, if the following conditions are satisfied:
    - $\mathscr{D}$ is relatively nef,
    - g is plurisubharmonic (see e.g. [26, Section 1.4]),
    - $h_{\overline{\mathscr{D}}}(x) \geq 0$ for all closed points $x \in X = \mathscr{X}_{\mathbb{Q}}$.
(4) $\overline{\mathscr{D}} \in \mathrm{ADiv}_{\mathbb{R}}(\mathscr{X})$ is called *integrable*, if it is in a real vector space generated by nef arithmetic $\mathbb{R}$-divisors of C^0-type. The space of integrable $\mathbb{R}$-divisors of C^0-type on $\mathscr{X}$ is denoted by $\mathrm{Div}^i_{\mathbb{R}}(\mathscr{X})$.
(5) $\overline{\mathscr{D}} \in \mathrm{ADiv}_{\mathbb{R}}(\mathscr{X})$ is called *ample*, if there exists a decomposition $\overline{\mathscr{D}} = \sum_{i=1}^m \alpha_i \overline{\mathscr{A}}_i$ with $m \geq 1$ and $\alpha_i > 0$ for $i = 1, \ldots, m$, where $\overline{\mathscr{A}}_i$ are arithmetic $\mathbb{Z}$-divisors of C^∞-type such that each $\mathscr{O}(\overline{\mathscr{A}}_i)$ is an ample Hermitian line bundle in the sense of [30, Section 4.1] (see also [25] for some discussion).

Let us remark that as the names suggest, there are some implications between the above introduced properties. For example, ample implies nef and big; and effective implies pseudo-effective (more generally, adding a pseudo-effective cannot decrease volume). Also, if $\overline{\mathscr{D}}$ is big, then $\mathscr{D}_{\mathbb{Q}}$ is big, and there is a natural number $n \geq 1$ and a rational function $f \in \kappa(\mathscr{X})^\times$ such that $\widehat{\mathrm{div}}(f) + n\overline{\mathscr{D}}$ is effective. For proofs of these facts see [25].

Lemma 2.2.11. Let $\overline{\mathscr{D}}$ be an arithmetic $\mathbb{R}$-divisor of C^0-type on $\mathscr{X}$. Fix $\varepsilon > 0$. Then there exist:

- ample arithmetic $\mathbb{R}$-divisors of C^∞-type $\overline{\mathscr{A}}, \overline{\mathscr{A}}'$;
- a C^0-function ϕ on $\mathscr{X}(\mathbb{C})$ with $\|\phi\|_{\sup} < \varepsilon$;

such that $\overline{\mathscr{D}}$ is rationally equivalent to $\overline{\mathscr{A}} - \overline{\mathscr{A}}' - (0, \phi)$.

Proof. By [25, Proposition 2.4.2] we can write $\overline{\mathscr{D}} = \sum_i a_i \overline{\mathscr{D}}_i$ for some arithmetic $\mathbb{Z}$-divisors of C^0-type $\overline{\mathscr{D}}_i$ and $a_i \in \mathbb{R}$. Let $\delta > 0$ be a real number and ϕ_i be a C^0-function on $\mathscr{X}(\mathbb{C})$ such that $\overline{\mathscr{D}}_i + (0, \phi_i)$ are of C^∞-type and with $\|\phi_i\|_{\sup} < \delta$. By [12, Corollary 2.5] we can present $\mathscr{O}(\overline{\mathscr{D}}_i + (0, \phi_i))$ as a difference of two ample Hermitian line bundles. By taking rational sections of those Hermitian line bundles, we can write $\overline{\mathscr{D}}_i + (0, \phi_i)$ as a difference of ample arithmetic $\mathbb{Z}$-divisors of C^∞-type

$\overline{\mathscr{A}}_i - \overline{\mathscr{A}}'_i$, at least up to a principal divisor. In particular, $\overline{\mathscr{D}}$ is rationally equivalent to

$$\sum_i a_i(\overline{\mathscr{A}}_i - \overline{\mathscr{A}}'_i) - (0, \sum_i a_i\phi_i).$$

Note that $\|\sum_i a_i\phi_i\|_{\sup} \leq \sum_i |a_i|\|\phi_i\|_{\sup} < (\sum_i |a_i|)\delta$. By taking δ small enough and rearranging the terms we get the result. □

2.3. **Intersection product and arithmetic volume.** We use a variant of arithmetic intersection theory defined in [25, 23]. In our notation an arbitrary arithmetic $\mathbb{R}$-divisor of C^0-type appears on the right in contrary to [23]. More precisely, there exists a symmetric multi-linear real-valued pairing $\widehat{\deg}(\overline{\mathscr{D}}_0 \cdots \overline{\mathscr{D}}_d)$ defined when $\overline{\mathscr{D}}_i$ are all integrable arithmetic $\mathbb{R}$-divisors of C^0-type, extending the intersection product of arithmetic $\mathbb{Z}$-divisors of C^∞-type (see e.g. [11] for the definition in that case). Moreover, one can extend it to the case where $\overline{\mathscr{D}}_0, \ldots, \overline{\mathscr{D}}_{d-1}$ are integrable, and $\overline{\mathscr{D}}_d$ is any arithmetic $\mathbb{R}$-divisor of C^0-type. This extension is the unique one having the following property: If $\overline{\mathscr{D}}_0, \ldots, \overline{\mathscr{D}}_{d-1}$ are nef and $\overline{\mathscr{D}}_d$ is pseudo-effective, then $\widehat{\deg}(\overline{\mathscr{D}}_0 \cdots \overline{\mathscr{D}}_{d-1} \cdot \overline{\mathscr{D}}_d) \geq 0$. Also, it only depends on the classes of the divisors in $\mathrm{APic}_{\mathbb{R}}(\mathscr{X})$ and is continuous in each variable separately. We sometimes write $\overline{\mathscr{D}}_0 \cdot \ldots \cdot \overline{\mathscr{D}}_d$ for $\widehat{\deg}(\overline{\mathscr{D}}_0 \cdots \overline{\mathscr{D}}_d)$ where $\overline{\mathscr{D}}_0, \ldots, \overline{\mathscr{D}}_{d-1}$ are integrable arithmetic $\mathbb{R}$-divisors of C^0-type and $\overline{\mathscr{D}}_d$ is an arithmetic $\mathbb{R}$-divisor of C^0-type.

Ikoma [23, Theorem 6.4] gave the following characterisation of pseudo-effectivity, in terms of the arithmetic intersection product.

Theorem 2.3.1. Let $\overline{\mathscr{D}}$ be an arithmetic $\mathbb{R}$-divisor of C^0-type on $\mathscr{X}$.

(1) The following are equivalent.
   (a) $\overline{\mathscr{D}}$ is pseudo-effective.
   (b) For any normalised blowup $\phi : \mathscr{X}' \to \mathscr{X}$ and for any nef arithmetic $\mathbb{R}$-divisor of C^0-type $\overline{\mathscr{H}}$ on $\mathscr{X}'$, we have
   $$\widehat{\deg}(\overline{\mathscr{H}}^d \cdot \phi^*\overline{\mathscr{D}}) \geq 0.$$
   (c) For any blowup $\phi : \mathscr{X}' \to \mathscr{X}$ such that $\mathscr{X}'$ is normal, generically smooth, and for any ample arithmetic $\mathbb{Q}$-divisor of C^∞-type $\overline{\mathscr{H}}$ on $\mathscr{X}'$, we have
   $$\widehat{\deg}(\overline{\mathscr{H}}^d \cdot \phi^*\overline{\mathscr{D}}) \geq 0.$$

(2) Suppose that $\overline{\mathscr{D}}$ is pseudo-effective. The following are equivalent.
   (a) $\overline{\mathscr{D}}$ is principal.
   (b) There exists a blowup $\phi : \mathscr{X}' \to \mathscr{X}$ with normal, generically smooth $\mathscr{X}'$, and an ample arithmetic $\mathbb{R}$-divisor of C^∞-type $\overline{\mathscr{H}}$ on $\mathscr{X}'$, such that
   $$\widehat{\deg}(\overline{\mathscr{H}}^d \cdot \phi^*\overline{\mathscr{D}}) = 0.$$

Definition 2.3.2. Let $\overline{\mathscr{D}}$ be a big arithmetic $\mathbb{R}$-divisor of C^0-type on $\mathscr{X}$. Let $\Theta(\overline{\mathscr{D}})$ be the set of pairs $(f : \mathscr{X}' \to \mathscr{X}, \overline{\mathscr{A}})$ consisting of a blowup $f : \mathscr{X}' \to \mathscr{X}$ together with a nef arithmetic $\mathbb{R}$-divisor of C^0-type on $\mathscr{X}'$ such that $f^*\overline{\mathscr{D}} - \overline{\mathscr{A}}$ is pseudo-effective. Let $\Theta_{\mathrm{amp}}(\overline{\mathscr{D}})$ be the set of pairs $(f : \mathscr{X}' \to \mathscr{X}, \overline{\mathscr{A}})$ in $\Theta(\overline{\mathscr{D}})$ where $\overline{\mathscr{A}}$ is an ample arithmetic $\mathbb{R}$-divisor of C^∞-type on $\mathscr{X}'$ and $f^*\overline{\mathscr{D}} - \overline{\mathscr{A}}$ is an effective arithmetic $\mathbb{Q}$-divisor of C^0-type on $\mathscr{X}'$. The *positive intersection d'th power of* $\overline{\mathscr{D}}$ is the function

$$\langle \overline{\mathscr{D}}^d \rangle : \mathrm{ADiv}_{\mathbb{R}}(\mathscr{X}) \to \mathbb{R}$$

defined in the following way:

$$\langle \overline{\mathscr{D}}^d \rangle \overline{\mathscr{E}} = \sup \Big\{ \overline{\mathscr{A}}^d \cdot f^*\overline{\mathscr{E}} : (f, \overline{\mathscr{A}}) \in \Theta_{\mathrm{amp}}(\overline{\mathscr{D}}) \Big\}$$

for big and nef $\overline{\mathscr{E}} \in \mathrm{ADiv}_{\mathbb{R}}(\mathscr{X})$, and for other $\overline{\mathscr{E}} \in \mathrm{ADiv}_{\mathbb{R}}(\mathscr{X})$ we extend by linearity and continuity (see [23] for more details).

One of the most important tools that we need is differentiability of arithmetic volume. It is a consequence of Yuan's arithmetic Siu's inequality [31], as noticed by Chen [14] (the geometric case was proved in [9]). In the context of arithmetic $\mathbb{R}$-divisors of C^0-type it appears e.g. in [23, Theorem 5.3].

Theorem 2.3.3. The function $\widehat{\mathrm{vol}} : \mathrm{ADiv}_{\mathbb{R}}(\mathscr{X}) \to \mathbb{R}$ has the following properties:

(1) It is continuous.

(2) It is positively $(d+1)$-homogeneous, i.e.,

$$\widehat{\mathrm{vol}}(t\overline{\mathscr{D}}) = t^{d+1}\widehat{\mathrm{vol}}(\overline{\mathscr{D}}),$$

for any $t > 0$ and arithmetic $\mathbb{R}$-divisor of C^0-type $\overline{\mathscr{D}}$ on $\mathscr{X}$.

(3) It is differentiable on the big cone and at a big $\overline{\mathscr{D}} \in \mathrm{ADiv}_{\mathbb{R}}(\mathscr{X})$ the derivative is given by

$$D_{\overline{\mathscr{D}}}\widehat{\mathrm{vol}}(\overline{\mathscr{E}}) = (d+1)\langle \overline{\mathscr{D}}^d \rangle \overline{\mathscr{E}}.$$

Lemma 2.3.4. Let $\overline{\mathscr{D}}_1, \dots, \overline{\mathscr{D}}_n$ be big and nef arithmetic $\mathbb{R}$-divisors of C^0-type on $\mathscr{X}$. Let $\overline{\mathscr{B}}$ be a big arithmetic $\mathbb{R}$-divisor of C^0-type on $\mathscr{X}$. Fix $\varepsilon > 0$. Then there is a pair $(f, \overline{\mathscr{A}}) \in \Theta_{\mathrm{amp}}(\overline{\mathscr{B}})$ such that for all $i = 1, \dots, n$

$$|\langle \overline{\mathscr{B}}^d \rangle \overline{\mathscr{D}}_i - \overline{\mathscr{A}}^d \cdot f^*\overline{\mathscr{D}}_i| < \varepsilon$$

and satisfying

$$\widehat{\mathrm{vol}}(\overline{\mathscr{B}}) \geq \widehat{\mathrm{vol}}(\overline{\mathscr{A}}) > \frac{1}{2}\widehat{\mathrm{vol}}(\overline{\mathscr{B}}).$$

Proof. First of all, we take $(f_0, \overline{\mathscr{A}}_0) \in \Theta(\overline{\mathscr{B}})$ satisfying

$$\widehat{\mathrm{vol}}(\overline{\mathscr{B}}) \geq \widehat{\mathrm{vol}}(\overline{\mathscr{A}}_0) > \frac{3}{4}\widehat{\mathrm{vol}}(\overline{\mathscr{B}}).$$

Such exists by the arithmetic Fujita approximation theorem [13], [32, Theorem C]; see also [23, Proposition 3.11.(1)]. Next, take pairs $(f_i, \overline{\mathscr{A}}_i) \in \Theta_{\mathrm{amp}}(\overline{\mathscr{B}})$ such that

$$\langle \overline{\mathscr{B}}^d \rangle \overline{\mathscr{D}}_i - \varepsilon \leq \overline{\mathscr{A}}_i^d \cdot f_i^*\overline{\mathscr{D}}_i \leq \langle \overline{\mathscr{B}}^d \rangle \overline{\mathscr{D}}_i$$

for all $i = 1, \dots, n$. By [23, Proposition 3.2] and induction, there is an element $(f : \mathscr{X}' \to \mathscr{X}, \overline{\mathscr{A}}') \in \Theta(\overline{\mathscr{D}})$ such that for all $i = 0, \dots, n$ there exists a factorisation

$$\mathscr{X}' \xrightarrow{g_i} \mathscr{X}_i \xrightarrow{f_i} \mathscr{X}$$

and

$$g_i^*\overline{\mathscr{A}}_i \leq \overline{\mathscr{A}}' \leq f^*\overline{\mathscr{D}}$$

with respect to the pseudo-effective cone. Choose a real number $0 < \gamma < 1$ to be specified later. By [23, Proposition 3.1] there is an ample $\overline{\mathscr{A}} \in \mathrm{ADiv}_{\mathbb{R}}(\mathscr{X}')$ such that $(f, \overline{\mathscr{A}}) \in \Theta_{\mathrm{amp}}(\overline{\mathscr{B}})$ and $\overline{\mathscr{A}} - \gamma\overline{\mathscr{A}}'$ is pseudo-effective. Thus we get:

$$\widehat{\mathrm{vol}}(\overline{\mathscr{B}}) \geq \widehat{\mathrm{vol}}(\overline{\mathscr{A}}) \geq \widehat{\mathrm{vol}}(\gamma\overline{\mathscr{A}}') = \gamma^d\widehat{\mathrm{vol}}(\overline{\mathscr{A}}') \geq \gamma^d\widehat{\mathrm{vol}}(\overline{\mathscr{A}}_0) > \gamma^d \cdot \frac{3}{4}\widehat{\mathrm{vol}}(\overline{\mathscr{B}}) > \frac{1}{2}\widehat{\mathrm{vol}}(\overline{\mathscr{B}}),$$

for γ such that $\gamma^d \cdot \frac{3}{4} > \frac{1}{2}$. Moreover for $i = 1, \dots, n$ we have

$$\langle \overline{\mathscr{B}}^d \rangle \overline{\mathscr{D}}_i \geq \overline{\mathscr{A}}^d \cdot f^* \overline{\mathscr{D}}_i \geq (\gamma \overline{\mathscr{A}}')^d \cdot f^* \overline{\mathscr{D}}_i \geq \gamma^d \cdot \overline{\mathscr{A}}_i^d \cdot f_i^* \overline{\mathscr{D}}_i$$
$$\geq \gamma^d (\langle \overline{\mathscr{B}}^d \rangle \overline{\mathscr{D}}_i - \varepsilon) = \gamma^d \langle \overline{\mathscr{B}}^d \rangle \overline{\mathscr{D}}_i - \gamma^d \varepsilon.$$

Thus

$$|\langle \overline{\mathscr{B}}^d \rangle \overline{\mathscr{D}}_i - \overline{\mathscr{A}}^d \cdot f^* \overline{\mathscr{D}}_i| \leq (1 - \gamma^d) \langle \overline{\mathscr{B}}^d \rangle \overline{\mathscr{D}}_i + \gamma^d \varepsilon.$$

Let $M := \max(\langle \overline{\mathscr{B}}^d \rangle \overline{\mathscr{D}}_i : i = 1, \dots, n)$. Since ε was arbitrary, by taking γ such that

$$(1 - \gamma^d) M + \gamma^d \varepsilon < 2\varepsilon,$$

we are done. □

2.4. **Essential infimum and Zhang's inequality.**

Definition 2.4.1. Let $\overline{\mathscr{D}}$ be an arithmetic $\mathbb{R}$-divisor of C^0-type on $\mathscr{X}$. The *essential infimum* of $\overline{\mathscr{D}}$ is the following number

$$\zeta(\overline{\mathscr{D}}) := \sup_{\mathscr{Y} \subset \mathscr{X}} \inf_{x \in \mathscr{X} \setminus \mathscr{Y}(\overline{\mathbb{Q}})} h_{\overline{\mathscr{D}}}(x),$$

where the supremum is taken taken over all closed strict subschemes $\mathscr{Y} \subset \mathscr{X}$. Note that ζ depends only on the rational class of an arithmetic $\mathbb{R}$-divisor of C^0-type and a priori has values in $\mathbb{R} \cup \{\pm\infty\}$.

Lemma 2.4.2. The essential infimum function ζ has the following properties:

(1) If $\overline{\mathscr{D}}$ is an arithmetic $\mathbb{R}$-divisor of C^0-type on $\mathscr{X}$ and $f : \mathscr{X}' \to \mathscr{X}$ is generically finite and surjective, then $\zeta(\overline{\mathscr{D}}) = \zeta(f^* \overline{\mathscr{D}})$.
(2) For any arithmetic $\mathbb{R}$-divisors of C^0-type $\overline{\mathscr{A}}, \overline{\mathscr{B}}$ on $\mathscr{X}$
$$\zeta(\overline{\mathscr{A}} + \overline{\mathscr{B}}) \geq \zeta(\overline{\mathscr{A}}) + \zeta(\overline{\mathscr{B}}).$$
(3) It is positively 1-homogeneous.
(4) It is non-negative on effective arithmetic $\mathbb{R}$-divisors of C^0-type.
(5) For any arithmetic $\mathbb{R}$-divisors of C^0-type $\overline{\mathscr{A}}, \overline{\mathscr{B}}$ on $\mathscr{X}$ such that $\overline{\mathscr{B}}$ is big and $\overline{\mathscr{A}} \leq \overline{\mathscr{B}}$ with respect to the pseudo-effective cone, we have $\zeta(\overline{\mathscr{A}}) \leq \zeta(\overline{\mathscr{B}})$.
(6) It has values in $\mathbb{R} \cup \{-\infty\}$.

Proof. The first property follows from Lemma 2.2.7. For the proof of properties two, three and four, see [2, Lemma 3.15]. For the fifth one fix arithmetic $\mathbb{R}$-divisors of C^0-type $\overline{\mathscr{A}}, \overline{\mathscr{B}}$ on $\mathscr{X}$ such that $\overline{\mathscr{B}}$ is big and $\overline{\mathscr{A}} \leq \overline{\mathscr{B}}$. Then we have

$$\zeta(\overline{\mathscr{B}}) = \lim_{\varepsilon \to 0^+} \zeta((1 + \varepsilon) \overline{\mathscr{B}}) = \lim_{\varepsilon \to 0^+} \zeta(\varepsilon \overline{\mathscr{B}} + (\overline{\mathscr{B}} - \overline{\mathscr{A}}) + \overline{\mathscr{A}})$$
$$\geq \liminf_{\varepsilon \to 0^+} \zeta(\varepsilon \overline{\mathscr{B}} + (\overline{\mathscr{B}} - \overline{\mathscr{A}})) + \zeta(\overline{\mathscr{A}}).$$

By pseudo-effectivity of $\overline{\mathscr{B}} - \overline{\mathscr{A}}$ and bigness of $\overline{\mathscr{B}}$ the divisor $\varepsilon \overline{\mathscr{B}} + (\overline{\mathscr{B}} - \overline{\mathscr{A}})$ is big, hence effective, and we are done by (4).

By Lemma 2.2.11 and the fact that values of ζ only depend on the rational equivalence classes of arithmetic $\mathbb{R}$-divisors of C^0-type, to get the last property it is enough to show that for an ample $\mathbb{Z}$-divisor of C^∞-type $\overline{\mathscr{A}}$ one has $\zeta(\overline{\mathscr{A}}) < \infty$. This follows from Zhang's inequalities, but for an elementary proof using Weil's height machine, see [8, Proposition 2.6]. □

Lemma 2.4.3. [2, Lemma 3.16] Let $\overline{\mathscr{D}}$ be an arithmetic $\mathbb{R}$-divisor of C^0-type on $\mathscr{X}$ and let $\overline{\mathscr{N}}$ be the zero divisor with Green's function 2. Then

(1) $\zeta(\overline{\mathscr{D}} - r \cdot \overline{\mathscr{N}}) = \zeta(\overline{\mathscr{D}}) - r$ for all real r;
(2) if $\overline{\mathscr{D}}$ is pseudo-effective and $\mathscr{D}_{\mathbb{Q}}$ is big, then $\zeta(\overline{\mathscr{D}}) \geq 0$;
(3) if $\mathscr{D}_{\mathbb{Q}}$ is big, the following inequality holds

$$\zeta(\overline{\mathscr{D}}) \geq \sup\{r : \overline{\mathscr{D}} - r \cdot \overline{\mathscr{N}} \text{ is pseudo-effective}\}.$$

Proof. The first point follows from additivity of height and Lemma 2.2.6. The second point is [2, Lemma 3.16]. The third point follows from the first two. □

Calculating essential infimum can be difficult, but Zhang [34] proved very powerful inequalities that give some control over it. We state the lower inequality, in this generality proved by Ballaÿ in [2, Theorem 7.2.(ii)].

Theorem 2.4.4. Let $\overline{\mathscr{D}}$ be a pseudo-effective arithmetic $\mathbb{R}$-divisor of C^0-type on $\mathscr{X}$ with $\mathscr{D}_{\mathbb{Q}}$ big. Then the following inequality holds

$$\zeta(\overline{\mathscr{D}}) \geq \frac{\widehat{\mathrm{vol}}(\overline{\mathscr{D}})}{(d+1)\,\mathrm{vol}(\mathscr{D}_{\mathbb{Q}})}.$$

2.5. **Convex geometry.**

Lemma 2.5.1. Let V be a finite dimensional real vector space and let V^+ be a closed convex cone in V not containing any line and with non-empty interior. Let $l : V \to \mathbb{R}$ be a linear functional non-negative on V^+. Then for any vectors $v_0, \dots, v_n \in V$ and for any $\varepsilon > 0$ there is a linear functional $l' : V \to \mathbb{R}$ strictly positive on $V^+ \setminus \{0\}$ such that for all $i = 0, \dots, n$

$$|l(v_i) - l'(v_i)| \leq \varepsilon.$$

Moreover, if $v_0 \in V^+$, then one can additionally assume that $l'(v_0) \geq l(v_0)$.

Proof. Fix $v_0, \dots, v_n \in V$ and $\varepsilon > 0$. It is enough to find a linear functional $m : V \to \mathbb{R}$ such that $V^+ \setminus \{0\} \subset \{v \in V : m(v) > 0\}$. Indeed, in this case we can just put $l' := l + \delta m$ for small enough positive δ. To define m one can equip V with a (non-degenerate, positively defined) scalar product and take m to be the product with a vector from the interior of the dual cone of V^+. □

Lemma 2.5.2. Let V be a finite dimensional real vector space and let V^+ be a closed convex cone in V generating the whole space. Assume that $l : V \to \mathbb{R}$ is a linear functional strictly positive on $V^+ \setminus \{0\}$ and that $\alpha : V \to \mathbb{R}$ is a non-negative function which is:

(1) continuous,
(2) positively 1-homogeneous,
(3) differentiable on $(V^+)^{\mathrm{o}}$,
(4) vanishing outside $(V^+)^{\mathrm{o}}$.

Then there is a unique continuous continuation of the function $\frac{\alpha}{l}$ to $V \setminus \{0\}$. Moreover, there exists a point $p \in (V^+)^{\mathrm{o}}$ such that

$$D_p\alpha = \frac{\alpha(p)}{l(p)} l.$$

Proof. Pick a Euclidean norm on V and let S be the unit sphere of it. Since both l and α are positively 1-homogeneous, for the first part it is enough to show that $\frac{\alpha}{l}$ has a unique continuous extension to S. This follows from the fact that if $l(q) = 0$ for $q \in S$, then $q \in S \cap V \setminus V^+$ and $S \cap V \setminus V^+$ is a relatively open set in S, where α is zero. To prove the second part, let $p \in S$ be a point where the maximum of $\frac{\alpha}{l}$ is achieved (it exists by compactness of S). Then $\alpha(p), l(p) > 0$, as otherwise α is the zero function. Point p is a local maximum of $\frac{\alpha}{l}$ and so also of $\log(\frac{\alpha}{l}) = \log(\alpha) - \log(l)$ (note that both functions are defined in a neighbourhood of p). Hence the derivative of $\log(\frac{\alpha}{l})$ at p vanishes and we get

$$\frac{D_p\alpha}{\alpha(p)} = D_p(\log(\alpha)) = D_p(\log(l)) = \frac{l}{l(p)}.$$

□

3. Applications in globally valued fields

3.1. **Approximating a global functional.** For this subsection we fix a normal, generically smooth arithmetic variety $\mathscr{X}$ over $\operatorname{Spec}(\mathbb{Z})$. Let $d+1 \geq 2$ be the dimension of $\mathscr{X}$.

Proposition 3.1.1. Let $\overline{\mathscr{B}}, \overline{\mathscr{D}}_0, \ldots, \overline{\mathscr{D}}_n$ be arithmetic $\mathbb{R}$-divisors of C^0-type on $\mathscr{X}$ with $\overline{\mathscr{B}}$ big (in particular $\zeta(\overline{\mathscr{B}}) > 0$). Fix $\varepsilon > 0$. Then there exists a blowup $f : \mathscr{X}' \to \mathscr{X}$ and an ample arithmetic $\mathbb{Q}$-divisor of C^∞-type $\overline{\mathscr{B}}' \leq f^*\overline{\mathscr{B}}$ on $\mathscr{X}'$ such that:

- $\operatorname{vol}(\mathscr{B}'_{\mathbb{Q}}) > \frac{\widehat{\operatorname{vol}}(\overline{\mathscr{B}})}{2(d+1)\zeta(\overline{\mathscr{B}})} =: A$;
- $|\langle \overline{\mathscr{B}}^d \rangle \overline{\mathscr{D}}_i - \overline{\mathscr{B}}'^d \cdot f^*\overline{\mathscr{D}}_i| < A\varepsilon$ for all $i = 0, \ldots, n$.

Proof. First, note that the expression

$$\langle \overline{\mathscr{B}}^d \rangle \overline{\mathscr{D}} - \overline{\mathscr{B}}'^d \cdot f^*\overline{\mathscr{D}}$$

is linear and continuous in $\overline{\mathscr{D}}$. Moreover, it only depends on the rational equivalence class of $\overline{\mathscr{D}}$. Hence, by Lemma 2.2.11 we can without loss of generality assume that $\overline{\mathscr{D}}_0, \ldots, \overline{\mathscr{D}}_n$ are ample arithmetic $\mathbb{R}$-divisors. In particular they are big and nef. Thus, by Lemma 2.3.4 for $A\varepsilon$ instead of ε, there is a pair $(f : \mathscr{X}' \to \mathscr{X}, \overline{\mathscr{A}}) \in \Theta_{\mathrm{amp}}(\overline{\mathscr{B}})$ such that for all $i = 0, \ldots, n$

$$|\langle \overline{\mathscr{B}}^d \rangle \overline{\mathscr{D}}_i - \overline{\mathscr{A}}^d \cdot f^*\overline{\mathscr{D}}_i| < A\varepsilon$$

and

$$\widehat{\operatorname{vol}}(\overline{\mathscr{B}}) \geq \widehat{\operatorname{vol}}(\overline{\mathscr{A}}) > \frac{1}{2}\widehat{\operatorname{vol}}(\overline{\mathscr{B}}).$$

We use Theorem 2.4.4 to get the middle inequality of

$$\frac{\widehat{\operatorname{vol}}(\overline{\mathscr{B}})}{2(d+1)\operatorname{vol}(\mathscr{A}_{\mathbb{Q}})} < \frac{\widehat{\operatorname{vol}}(\overline{\mathscr{A}})}{(d+1)\operatorname{vol}(\mathscr{A}_{\mathbb{Q}})} \leq \zeta(\overline{\mathscr{A}}) \leq \zeta(f^*\overline{\mathscr{B}}) = \zeta(\overline{\mathscr{B}}).$$

The last inequality is by Lemma 2.4.2. We get

$$\operatorname{vol}(\mathscr{A}_{\mathbb{Q}}) > \frac{\widehat{\operatorname{vol}}(\overline{\mathscr{B}})}{2(d+1)\zeta(\overline{\mathscr{B}})} = A.$$

Note that $\overline{\mathscr{A}}$ is an ample arithmetic $\mathbb{R}$-divisor of C^∞-type on $\mathscr{X}'$, so it is of the form

$$\overline{\mathscr{A}} = \sum_j a_j \overline{\mathscr{A}}_j,$$

for some some $a_j \in \mathbb{R}$, and $\overline{\mathscr{A}}_j$ being ample arithmetic $\mathbb{Z}$-divisors of C^∞-type. Replacing a_j's by sufficiently good rational approximations, we can find an ample arithmetic $\mathbb{Q}$-divisor of C^∞-type $\overline{\mathscr{B}}'$ on $\mathscr{X}'$ satisfying the desired inequalities. This is because of multilinearity of the arithmetic intersection product and the continuity of the volume vol. $\square$

The below proposition crucially relies on arithmetic Bertini theorems proved by Charles [12] and Wilms [30]. Also, for our application the arithmetic Demailly approximation theorem proved by Qu and Yin [27] would suffice.

Proposition 3.1.2. Let $\overline{\mathscr{B}}, \overline{\mathscr{D}}_0, \dots, \overline{\mathscr{D}}_n$ be arithmetic $\mathbb{R}$-divisors of C^0-type on $\mathscr{X}$ with $\overline{\mathscr{B}}$ being an ample arithmetic $\mathbb{Q}$-divisor of C^∞-type. Let $Z \subset X = \mathscr{X} \otimes \mathbb{Q}$ be a closed strict subscheme. Fix $\varepsilon > 0$. Then there exists a closed point $x \in X \setminus Z$ such that

$$\left| h_{\overline{\mathscr{D}}_i}(x) - \frac{\overline{\mathscr{B}}^d \cdot \overline{\mathscr{D}}_i}{\mathrm{vol}(\mathscr{B}_{\mathbb{Q}})} \right| < \varepsilon, \text{ for all } i = 0, \dots, n.$$

Proof. First, note that $\frac{\overline{\mathscr{B}}^d \cdot \overline{\mathscr{D}}_i}{\mathrm{vol}(\mathscr{B}_{\mathbb{Q}})}$ does not change when we replace $\overline{\mathscr{B}}$ with a multiple of itself, so we can without loss of generality assume that $\overline{\mathscr{B}}$ is an ample $\mathbb{Z}$-arithmetic divisor of C^∞-type. Also, the expression

$$h_{\overline{\mathscr{D}}}(x) - \frac{\overline{\mathscr{B}}^d \cdot \overline{\mathscr{D}}}{\mathrm{vol}(\mathscr{B}_{\mathbb{Q}})}$$

is linear in $\overline{\mathscr{D}}$ and depends only on its rational equivalence class. Thus, by continuity in $\overline{\mathscr{D}}$ and Lemma 2.2.11, we can without loss of generality assume that all $\overline{\mathscr{D}}_i$'s are ample arithmetic $\mathbb{Z}$-divisors of C^∞-type.

The above reduction allows us to work with ample Hermitian line bundles

$$\mathscr{O}(\overline{\mathscr{B}}), \mathscr{O}(\overline{\mathscr{D}}_0), \dots, \mathscr{O}(\overline{\mathscr{D}}_n) \text{ on } \mathscr{X}.$$

To ease the notation we do not write $\mathscr{O}(-)$ for the corresponding Hermitian line bundles. We need the following.

Claim 3.1.3 (Global Bertini)**.** Let $\mathscr{W} \subset \mathscr{X}$ be a generically smooth arithmetic subvariety (not necessarily normal) of dimension $k+1$ for $k \geq 1$, that is not contained in the closure of Z in $\mathscr{X}$. Then, there is a natural $m \geq 1$ and a small section s of $(\overline{\mathscr{B}}|_{\mathscr{W}})^{\otimes m}$ such that:

(1) The effective divisor $\mathrm{div}(s) \subset \mathscr{W}$ is an irreducible, generically smooth arithmetic variety not contained in the closure of Z.
(2) We have $\left|\widehat{\deg}(\overline{\mathscr{D}}_i \cdot \overline{\mathscr{B}}^{k-1} | \mathrm{div}(s)) - \frac{1}{m}\widehat{\deg}(\overline{\mathscr{D}}_i \cdot \overline{\mathscr{B}}^k | \mathscr{W})\right| < \varepsilon$ for $i = 0, \dots, n$.

Proof. Note that $\overline{\mathscr{B}}|_{\mathscr{W}}$ is ample, for example by [12, Corollary 2.8]. By [30, Remark 6.7.(ii)] together with [12, Theorem 2.21] taking a (uniformly) random small section of $(\overline{\mathscr{B}}|_{\mathscr{W}})^{\otimes m}$, for m big enough, will work with probability converging to one. $\square$

Using the claim d times, we find small sections $s_1, \dots, s_d$ of

$$\overline{\mathscr{B}}^{\otimes n_1}, (\overline{\mathscr{B}}|_{\mathrm{div}(s_1)})^{\otimes n_2}, \dots, (\overline{\mathscr{B}}|_{\mathrm{div}(s_1) \cap \ldots \cap \mathrm{div}(s_{d-1})})^{\otimes n_d}$$

respectively, such that the following hold:

(1) $\mathrm{div}(s_1) \cap \cdots \cap \mathrm{div}(s_d)$ is an irreducible, generically smooth arithmetic variety of dimension 1, hence it is of the form $\overline{\{x\}}$ for some closed point $x \in X$. Moreover, we can assume that $x \notin Z$.

(2) For all $i = 0, \dots, n$ we have

$$(\clubsuit) \qquad \left| \overline{\mathscr{D}}_i \cdot \overline{\mathscr{B}}^d - \frac{1}{n_1 \cdots n_d} \widehat{\deg}(\overline{\mathscr{D}}_i | \overline{\{x\}}) \right| < d\varepsilon.$$

By the inductive definition of the ordinary intersection product of line bundles on a variety X (see for example [11, Remark (2.6)]) and by ampleness of $\mathscr{B}_{\mathbb{Q}}$ we get

$$\begin{aligned}
\operatorname{vol}(\mathscr{B}_{\mathbb{Q}}) = \mathscr{B}_{\mathbb{Q}}^d &= \frac{1}{n_1} \deg(\mathscr{B}_{\mathbb{Q}}^{d-1} | \operatorname{div}(s_1)_{\mathbb{Q}}) \\
&= \frac{1}{n_1 \cdot n_2} \deg(\mathscr{B}_{\mathbb{Q}}^{d-2} | \operatorname{div}(s_1)_{\mathbb{Q}} \cap \operatorname{div}(s_2)_{\mathbb{Q}}) \\
= \cdots &= \frac{1}{n_1 \cdots n_d} \deg(\operatorname{div}(s_1)_{\mathbb{Q}} \cap \cdots \cap \operatorname{div}(s_d)_{\mathbb{Q}}) \\
&= \frac{1}{n_1 \cdots n_d} \deg(x).
\end{aligned}$$

Hence, using Equation ($\clubsuit$) we end up with

$$\left| \frac{\overline{\mathscr{D}}_i \cdot \overline{\mathscr{B}}^d}{\operatorname{vol}(\mathscr{B}_{\mathbb{Q}})} - \frac{\widehat{\deg}(\overline{\mathscr{D}}_i | \overline{\{x\}}) |}{\deg(x)} \right| < \frac{d}{\operatorname{vol}(\mathscr{B}_{\mathbb{Q}})} \varepsilon \text{ for all } i = 0, \dots, n.$$

By the definition of height we have

$$h_{\overline{\mathscr{D}}_i}(x) = \frac{\widehat{\deg}(\overline{\mathscr{D}}_i | \overline{\{x\}})}{\deg(x)},$$

so

$$\left| h_{\overline{\mathscr{D}}_i}(x) - \frac{\overline{\mathscr{D}}_i \cdot \overline{\mathscr{B}}^d}{\operatorname{vol}(\mathscr{B}_{\mathbb{Q}})} \right| < \frac{d}{\operatorname{vol}(\mathscr{B}_{\mathbb{Q}})} \varepsilon \text{ for all } i = 0, \dots, n.$$

This finishes the proof, as ε was arbitrary. □

For a closed point $x \in X$ consider the function $h_{(-)}(x) : \operatorname{APic}_{\mathbb{R}}(\mathscr{X}) \to \mathbb{R}$ taking $\overline{\mathscr{D}}$ to $h_{\overline{\mathscr{D}}}(x)$. It is linear, non-negative on classes of effective arithmetic $\mathbb{R}$-divisors of C^0-type, and takes the zero divisor with Green's function 2, to 1. Denote the set of all functions $l : \operatorname{APic}_{\mathbb{R}}(\mathscr{X}) \to \mathbb{R}$ having these properties by $\operatorname{APic}_{\mathbb{R}}(\mathscr{X})^{\vee}_{+,1}$ and equip it with the weak*-topology. Now we are ready to prove Theorem B (as mentioned in Remark 2.1.2, normality of $\mathscr{X}$ can be assumed for free by passing to a blowup).

Theorem 3.1.4. Fix a closed strict subscheme $Z \subset X = \mathscr{X} \otimes \mathbb{Q}$. Functions of the form $h_{(-)}(x)$, for closed points $x \in X \setminus Z$, are dense in $\operatorname{APic}_{\mathbb{R}}(\mathscr{X})^{\vee}_{+,1}$.

Proof. Let $\overline{\mathscr{D}}_0, \dots, \overline{\mathscr{D}}_n$ be arithmetic $\mathbb{R}$-divisors of C^0-type on $\mathscr{X}$ with $\overline{\mathscr{D}}_0$ being the zero divisor with the Green's function 2. Fix $\varepsilon > 0$ and $l \in \operatorname{APic}_{\mathbb{R}}(\mathscr{X})^{\vee}_{+,1}$. We will find a closed point $x \in X \setminus Z$ such that for all $i = 0, \dots, n$:

$$|h_{\overline{\mathscr{D}}_i}(x) - l(\overline{\mathscr{D}}_i)| < \varepsilon.$$

Put $M := \max(|l(\overline{\mathscr{D}}_0)|, \dots, |l(\overline{\mathscr{D}}_n)|)$ and let V be the real span of classes of $\overline{\mathscr{D}}_0, \dots, \overline{\mathscr{D}}_n, \overline{\mathscr{M}}$ in $\operatorname{APic}_{\mathbb{R}}(\mathscr{X})$, where $\overline{\mathscr{M}}$ is some big arithmetic $\mathbb{R}$-divisor of C^0-type in $\operatorname{ADiv}_{\mathbb{R}}(\mathscr{X})$. Let V^+ be the (Euclidean) closure of the subset of classes of big arithmetic $\mathbb{R}$-divisors of C^0-type from V. It is a convex closed cone that does not contain any line in V by Theorem 2.3.1. Moreover, $\overline{\mathscr{D}}_0$ is effective, hence it lies in V^+. Thus, by Lemma 2.5.1 we can approximate l (by the assumptions it is well defined on V) arbitrarily close by linear functionals strictly positive on $V^+ \setminus \{0\}$ and non-decreasing the value on $\overline{\mathscr{D}}_0$. Since ε is arbitrary in the statement of the theorem, we can without loss of generality replace l with sufficiently good approximation, so we assume that l is strictly positive on $V^+ \setminus \{0\}$, $l(\overline{\mathscr{D}}_0) = 1 + \varepsilon'$ for some $0 \leq \varepsilon' < \varepsilon$, and that $\max(|l(\overline{\mathscr{D}}_0)|, \ldots, |l(\overline{\mathscr{D}}_n)|) < 2M$.

We use Lemma 2.5.2 with $\alpha = \widehat{\mathrm{vol}}^{1/(d+1)}$ and V^+, V, l as above. Note that the assumptions of the lemma are satisfied by Theorem 2.3.3 which implies that $(V^+)^{\mathrm{o}}$ is the set of classes of big arithmetic $\mathbb{R}$-divisors of C^0-type in V. Thus there exists $p \in (V^+)^{\mathrm{o}}$ such that

$$\frac{D_p \alpha}{\alpha(p)} = \frac{l}{l(p)}.$$

If $\overline{\mathscr{B}}$ represents the class of p, then by Theorem 2.3.3 we get

$$\frac{\langle \overline{\mathscr{B}}^d \rangle}{\widehat{\mathrm{vol}}(\overline{\mathscr{B}})} = \frac{l}{l(\overline{\mathscr{B}})}.$$

By Proposition 3.1.1 there is a blowup $f : \mathscr{X}' \to \mathscr{X}$ and an ample arithmetic $\mathbb{Q}$-divisor of C^∞-type $\overline{\mathscr{B}}'$ on $\mathscr{X}'$ such that the following hold:

- $\mathrm{vol}(\mathscr{B}'_{\mathbb{Q}}) > \frac{\widehat{\mathrm{vol}}(\overline{\mathscr{B}})}{2(d+1)\zeta(\overline{\mathscr{B}})} =: A$;
- $|\langle \overline{\mathscr{B}}^d \rangle \overline{\mathscr{D}}_i - \overline{\mathscr{B}}'^d \cdot f^* \overline{\mathscr{D}}_i| < A\varepsilon$ for all $i = 1, \ldots, n$.

Now, we can use Proposition 3.1.2 for the ample arithmetic $\mathbb{Q}$-divisor of C^∞-type $\overline{\mathscr{B}}'$ and arithmetic $\mathbb{R}$-divisors of C^0-type $f^*\overline{\mathscr{D}}_0, \ldots, f^*\overline{\mathscr{D}}_n$ to find a closed point $x' \in X' \setminus f^{-1}(Z)$ (where $X' := \mathscr{X}' \otimes \mathbb{Q}$) which satisfies

$$\left| h_{f^*\overline{\mathscr{D}}_i}(x') - \frac{\overline{\mathscr{B}}'^d \cdot f^*\overline{\mathscr{D}}_i}{\mathrm{vol}(\mathscr{B}'_{\mathbb{Q}})} \right| < \varepsilon \text{ for all } i = 0, \ldots, n.$$

By putting the above equations and inequalities together we get that for any natural i smaller or equal n we have

$$(\diamondsuit) \qquad \left| h_{f^*\overline{\mathscr{D}}_i}(x') - \frac{\widehat{\mathrm{vol}}(\overline{\mathscr{B}})}{\mathrm{vol}(\mathscr{B}'_{\mathbb{Q}}) l(\overline{\mathscr{B}})} \cdot l(\overline{\mathscr{D}}_i) \right| < \Big(1 + \frac{A}{\mathrm{vol}(\mathscr{B}'_{\mathbb{Q}})}\Big)\varepsilon < 2\varepsilon.$$

Let $C := \frac{\widehat{\mathrm{vol}}(\overline{\mathscr{B}})}{\mathrm{vol}(\mathscr{B}'_{\mathbb{Q}}) l(\overline{\mathscr{B}})}$. Applying the last inequality for $i = 0$ we get

$$|1 - C(1+\varepsilon')| = \left| h_{f^*\overline{\mathscr{D}}_0}(x') - C \cdot l(\overline{\mathscr{D}}_0) \right| < 2\varepsilon,$$

because of Lemma 2.2.6. Thus we get that

$$|C - 1| \leq \left| C - \frac{1}{1+\varepsilon'} \right| + \left| \frac{1}{1+\varepsilon'} - 1 \right| < \frac{2\varepsilon}{1+\varepsilon'} + \frac{\varepsilon'}{1+\varepsilon'} \leq 3\varepsilon.$$

Recall that $\max(|l(\overline{\mathscr{D}}_0)|, \ldots, |l(\overline{\mathscr{D}}_n)|) < 2M$. By using the bound on $|C - 1|$ in Equation $(\diamondsuit)$ we get

$$\left| h_{f^*\overline{\mathscr{D}}_i}(x') - l(\overline{\mathscr{D}}_i) \right| < \big(2 + 6M\big)\varepsilon.$$

Let $x = f(x') \in X$. By Lemma 2.2.7 we have $h_{f^*\overline{\mathscr{D}}_i}(x') = h_{\overline{\mathscr{D}}_i}(x)$. Since M is independent of ε (it only depends on the initial data), this finishes the proof of the theorem. $\square$

3.2. **Lattice divisors and globally valued field functionals.** Fix a finitely generated field extension $\mathbb{Q} \subset F$ for this subsection.

Definition 3.2.1. We call $\mathscr{X}$ an *F-model*, if it is a generically smooth, normal arithmetic variety together with an isomorphism $\kappa(\mathscr{X}) \simeq F$. The system of F-models together with morphisms respecting isomorphisms of function fields with F is cofiltered (i.e., any two F-models can be dominated by a third). We use the following notations for direct limits of various groups of divisors with respect to this filtering

$$\mathrm{ADiv}_{\mathbb{K}}(F) := \varinjlim \mathrm{ADiv}_{\mathbb{K}}(\mathscr{X}) \text{ for } \mathbb{K} = \mathbb{Z}, \mathbb{Q} \text{ or } \mathbb{R};$$
$$\mathrm{RDiv}_{\mathbb{K}}(F) := \varinjlim \mathrm{RDiv}_{\mathbb{K}}(\mathscr{X}) \text{ for } \mathbb{K} = \mathbb{Z}, \mathbb{Q} \text{ or } \mathbb{R};$$
$$\mathrm{APic}_{\mathbb{R}}(F) := \varinjlim \mathrm{APic}_{\mathbb{R}}(\mathscr{X}).$$

We write $\mathrm{ADiv}^0_{\mathbb{K}}(\mathscr{X})$ for the group of arithmetic $\mathbb{K}$-divisors of C^0-type such that the underlying divisor is 0. This space can be naturally identified with the set of C^0-type functions on $\mathscr{X}(\mathbb{C})$. We define

$$\mathrm{ADiv}^0_{\mathbb{K}}(F) := \varinjlim \mathrm{ADiv}^0_{\mathbb{K}}(\mathscr{X}) \text{ for } \mathbb{K} = \mathbb{Z}, \mathbb{Q} \text{ or } \mathbb{R}.$$

Note that $\mathrm{ADiv}^0_{\mathbb{K}}(F)$ has a norm induced by the supremum norms on C^0-functions on (analytifications of) F-models. We use the notation $\|\cdot\|_{\sup}$ for this norm.

Remark 3.2.2. Note that there is an exact sequence

$$0 \longrightarrow \mathrm{RDiv}_{\mathbb{R}}(F) \longrightarrow \mathrm{ADiv}_{\mathbb{R}}(F) \longrightarrow \mathrm{APic}_{\mathbb{R}}(F) \longrightarrow 0$$

induced by similar exact sequences on all F-models $\mathscr{X}$.

Next we introduce a lattice structure on the space of arithmetic divisors on F-models. This structure has also been used in [29, 28] using the language of (generalised or b-) Weil functions. For effective Cartier divisors $\mathscr{D}, \mathscr{E}$ on $\mathscr{X}$ we write $\mathscr{C} \cap \mathscr{D}$ for their scheme-theoretic intersection, that is, for the closed subscheme whose ideal sheaf is generated by the local equations of $\mathscr{C}$ and $\mathscr{D}$.

Definition 3.2.3. Let $\overline{\mathscr{D}}, \overline{\mathscr{E}} \in \mathrm{ADiv}_{\mathbb{Q}}(F)$ and assume that $\mathscr{X}$ is an F-model such that $\overline{\mathscr{D}}, \overline{\mathscr{E}} \in \mathrm{ADiv}_{\mathbb{Q}}(\mathscr{X})$. We define the *join* $\overline{\mathscr{D}} \wedge \overline{\mathscr{E}}$ of $\overline{\mathscr{D}} = (\mathscr{D}, g)$ and $\overline{\mathscr{E}} = (\mathscr{E}, h)$ in the following way.

(1) Assume that $\mathscr{D}, \mathscr{E}$ are effective Cartier divisors on $\mathscr{X}$ and that $\mathscr{D} \cap \mathscr{E}$ is a Cartier divisor. Then we put $\overline{\mathscr{D}} \wedge \overline{\mathscr{E}} := (\mathscr{D} \cap \mathscr{E}, \min(g, h))$ as an element of $\mathrm{ADiv}_{\mathbb{Q}}(F)$.
(2) Assume that $\mathscr{D}, \mathscr{E}$ are effective Cartier divisors on $\mathscr{X}$. Let $\mathscr{X}'$ be an F-model over $\mathscr{X}$ such that the intersection of the pullbacks of $\mathscr{D}, \mathscr{E}$ to $\mathscr{X}'$ is a Cartier divisor. Then use the above definition on $\mathscr{X}'$.
(3) Assume that $\mathscr{D}, \mathscr{E}$ are Cartier divisors on $\mathscr{X}$. By [25, Lemma 5.2.3] there exists an effective Cartier divisor $\mathscr{A}$ on $\mathscr{X}$ such that $\mathscr{D} + \mathscr{A}, \mathscr{E} + \mathscr{A}$ are effective. Equip $\mathscr{A}$ with an $\mathscr{A}$-Green's function a on $\mathscr{X}$. Define $\overline{\mathscr{D}} \wedge \overline{\mathscr{E}} := ((\overline{\mathscr{D}} + \overline{\mathscr{A}}) \wedge (\overline{\mathscr{E}} + \overline{\mathscr{A}})) - \overline{\mathscr{A}}$ for $\overline{\mathscr{A}} = (\mathscr{A}, a)$.
(4) Assume that $\mathscr{D}, \mathscr{E}$ are $\mathbb{Q}$-Cartier divisors on $\mathscr{X}$. Let n be a natural number such that $n\mathscr{D}, n\mathscr{E}$ are Cartier divisors on $\mathscr{X}$. Put $\overline{\mathscr{D}} \wedge \overline{\mathscr{E}} := \frac{1}{n}(n\overline{\mathscr{D}} \wedge n\overline{\mathscr{E}})$.

If $\mathscr{D}, \mathscr{E}$ are $\mathbb{Q}$-Cartier divisors on $\mathscr{X}$, we define $\mathscr{D} \wedge \mathscr{E}$ using the same procedure.

Lemma 3.2.4. The join operation on $\mathrm{ADiv}_{\mathbb{Q}}(F)$ is well defined and does not depend on the choices in the definition.

Proof. Fix the notation from Definition 3.2.3. We check point by point, that the procedure does not depend on the choices made.

(1) Assume that $\mathscr{D},\mathscr{E}$ are effective Cartier divisors on $\mathscr{X}$ such that $\mathscr{D}\cap\mathscr{E}$ is an effective Cartier divisor. We need to show that $\min(g,h)$ is a $(\mathscr{D}\cap\mathscr{E})$-Green's function on $\mathscr{X}$. Let $\operatorname{Spec} A\subset\mathscr{X}$ be an open subset where $\mathscr{D},\mathscr{E},\mathscr{D}\cap\mathscr{E}$ are given respectively by d,e,f and assume that $x,y,d',e'\in A$ are such that $d=fd', e=fe', f=xd+ye$. Note that $f\cdot(1-(xd'+ye'))=0$, so $f=0$ or $xd'+ye'=1$, as A is an integral domain. If $f=0$, then $d=e=0$ and this cannot happen. We must show that the function

$$\min(g(p),h(p))+\log|f|^2(p)$$

extends to a C^0-type function on $(\operatorname{Spec} A)(\mathbb{C})$. By the assumption we know that functions

$$g(p)+\log|d|^2(p) \text{ and } h(p)+\log|e|^2(p)$$

extend to a C^0-type functions on $(\operatorname{Spec} A)(\mathbb{C})$. Note that

$$g(p)+\log|d|^2(p)=g(p)+\log|f|^2(p)+\log|d'|^2(p),$$
$$h(p)+\log|e|^2(p)=h(p)+\log|f|^2(p)+\log|e'|^2(p).$$

Thus it is enough to show that for every $p\in(\operatorname{Spec} A)(\mathbb{C})$ we have $|d'|(p)\neq 0$ or $|e'|(p)\neq 0$. However, $xd'+ye'=1$, so if $d'(p)=e'(p)=0$, then $0=x(p)d'(p)+y(p)e'(p)=1$ which gives a contradiction.

(2) Assume that $\mathscr{D},\mathscr{E}$ are effective Cartier divisors on $\mathscr{X}$ and let $\mathscr{X}',\mathscr{X}''$ be F-models over $\mathscr{X}$ where intersections of pullbacks of $\mathscr{D},\mathscr{E}$ are Cartier divisors. Without loss of generality $\mathscr{X}''$ is over $\mathscr{X}'$. Consider open sets $\operatorname{Spec} C\to\operatorname{Spec} B\to\operatorname{Spec} A$ inside $\mathscr{X}''\to\mathscr{X}'\to\mathscr{X}$ respectively, and assume that $\mathscr{D},\mathscr{E}$ have local equations $d,e\in A$ on $\operatorname{Spec}(A)$. Assume also, that on $\operatorname{Spec} B$ we have $(d,e)=(f)$ for some $f\in B$. Let $x,y,d',e'\in B$ be elements such that $d=fd',e=fe',f=xd+ye$. The same equations hold in C (for pullbacks), so (d,e) calculated in $\operatorname{Spec} B$ and pulled back to $\operatorname{Spec} C$ is the same as (d,e) calculated in $\operatorname{Spec} C$. Note that to define the Green's function for the intersection of pullbacks of $\mathscr{D},\mathscr{E}$ on $\mathscr{X}''$ one can take the minimum of pullbacks of g,h to $\mathscr{X}''$, or the pullback of the minimum of pullbacks of g,h on $\mathscr{X}'$.

(3) Assume that $\mathscr{D},\mathscr{E}$ are Cartier divisors on $\mathscr{X}$ and let $\mathscr{A},\mathscr{B}$ be effective Cartier divisors on $\mathscr{X}$ with $\mathscr{D}+\mathscr{A},\mathscr{E}+\mathscr{A}$ effective. Replace $\mathscr{X}$ by an F-model $\mathscr{X}'$ over $\mathscr{X}$ where $(\mathscr{D}+\mathscr{A})\cap(\mathscr{E}+\mathscr{A})$ is an effective Cartier divisor. To prove independence of $\overline{\mathscr{A}}$ in the definition of the lattice infimum (point 3.) it is enough to show that for any effective Cartier divisor $\mathscr{B}$ on $\mathscr{X}'$ we have

$$((\mathscr{D}+\mathscr{A})\cap(\mathscr{E}+\mathscr{A}))-\mathscr{A}=((\mathscr{D}+\mathscr{A}+\mathscr{B})\cap(\mathscr{E}+\mathscr{A}+\mathscr{B}))-(\mathscr{A}+\mathscr{B}).$$

To ease the notation, replace $\mathscr{D}+\mathscr{A},\mathscr{E}+\mathscr{A}$ by $\mathscr{D},\mathscr{E}$ respectively. We want to prove that $\mathscr{D}\cap\mathscr{E}=((\mathscr{D}+\mathscr{B})\cap(\mathscr{E}+\mathscr{B}))-\mathscr{B}$ assuming that $\mathscr{D},\mathscr{E},\mathscr{D}\cap\mathscr{E},\mathscr{B}$ are effective Cartier divisors on $\mathscr{X}$. Let $\operatorname{Spec} A\subset\mathscr{X}$ be an open subset where the effective Cartier divisors from the previous sentence are given respectively by $d,e,f,b\in A$. Also, assume that $\operatorname{Spec} A$ is chosen so that there are $x,y,d',e'\in A$ such that $d=fd',e=fe',f=xd+ye$. Note that then $bd=bfd',be=bfe',bf=xbd+ybe$, hence $(bf)=(be,bd)$, which gives the desired result. The fact that the $(\mathscr{D}\cap\mathscr{E})$-Green's function on $\mathscr{X}$ does not depend on $\overline{\mathscr{A}}$ follows from the fact that for real-valued functions g,h,j we have $\min(g+j,h+j)=\min(g,h)+j$.

(4) Finally, assume that $\mathscr{D},\mathscr{E}$ are $\mathbb{Q}$-Cartier and let $n\neq 0$ be a natural number such that $n\mathscr{D},n\mathscr{E}$ are Cartier divisors on $\mathscr{X}$. Fix a natural number $m\neq 0$. We need to show that using the point 3. of Definition 3.2.3 we get $(n\mathscr{D})\wedge(n\mathscr{E})=\frac{1}{m}((mn\mathscr{D})\wedge(mn\mathscr{E}))$. Replace $\mathscr{D},\mathscr{E}$ by $n\mathscr{D},n\mathscr{E}$ respectively, to assume that $n=1$. By the previous points in this lemma, it is enough to prove that if $\mathscr{D},\mathscr{E}$ are effective Cartier divisors on $\mathscr{X}$ with $\mathscr{D}\cap\mathscr{E}$ effective Cartier, then $(m\mathscr{D})\cap(m\mathscr{E})=m(\mathscr{D}\cap\mathscr{E})$. Let $\operatorname{Spec}A\subset\mathscr{X}$ be an open subset where $\mathscr{D},\mathscr{E},\mathscr{D}\cap\mathscr{E}$ are given respectively by d,e,f and assume that $x,y,d',e'\in A$ are such that $d=fd',e=fe',1=xd'+ye'$ (see point 1. in this lemma). Note that

$$1=(xd'+ye')^{2m}=\sum_{i=0}^{2m}\binom{2m}{i}x^iy^{2m-i}(d')^i(e')^{2m-i}\in((d')^m,(e')^m).$$

Thus, from the fact that $d^m=f^m(d')^m,e^m=f^m(e')^m$, we get that $(d^m,e^m)=(f^m)$, which finishes the proof. The fact that the $(\mathscr{D}\cap\mathscr{E})$-Green's function on $\mathscr{X}$ does not depend on m follows from the fact that $\min(mg,mh)=m\min(g,h)$.

□

Recall that in Definition 2.2.2 we have introduced a pairing $\beta(v,\overline{\mathscr{D}})$ between valuations v on F and arithmetic $\mathbb{R}$-divisors $\overline{\mathscr{D}}\in\operatorname{ADiv}_{\mathbb{R}}(\mathscr{X})$ on an F-model $\mathscr{X}$.

Lemma 3.2.5. Let v be a valuation on F. If $\overline{\mathscr{D}}\in\operatorname{ADiv}_{\mathbb{R}}(F)$, then the value $\beta(v,\overline{\mathscr{D}})$ does not depend on the F-model on which $\overline{\mathscr{D}}$ is defined. Moreover, the function $\beta(v,-)$ has the following properties:

(1) it is $\mathbb{R}$-linear,
(2) for every $\overline{\mathscr{D}},\overline{\mathscr{E}}\in\operatorname{ADiv}_{\mathbb{Q}}(F)$ we have $\beta(v,\overline{\mathscr{D}}\wedge\overline{\mathscr{E}})=\min(\beta(v,\overline{\mathscr{D}}),\beta(v,\overline{\mathscr{E}}))$.

Proof. The fact that $\beta(v,\overline{\mathscr{D}})$ does not depend on the F-model follows from the definition of pullback, see Remark 2.1.6. Linearity is clear. To prove the second point, by Lemma 3.2.4 and $\mathbb{R}$-linearity we can reduce to the following situation. Assume that $\overline{\mathscr{D}},\overline{\mathscr{E}}$ are arithmetic $\mathbb{Z}$-divisors on $\mathscr{X}$ with $\mathscr{D},\mathscr{E},\mathscr{D}\cap\mathscr{E}$ effective Cartier. If v is Archimedean, the statement follows from the definition, so assume that v is non-Archimedean. Let $p=\operatorname{supp}(v)\in\mathscr{X}$ and let $\operatorname{Spec}A\subset\mathscr{X}$ be an open neighbourhood of p where $\mathscr{D},\mathscr{E},\mathscr{D}\cap\mathscr{E}$ are given by $d,e,f\in A$ respectively. Assume additionally that $x,y,d',e'\in A$ are such that $d=fd',e=fe',1=xd'+ye'$ (see point 1. in Lemma 3.2.4). We need to show that $v(f)=\min(v(d),v(e))$. Note that

$$0=v(1)=v(xd'+ye')\geq\min(v(x)+v(d'),v(y)+v(e'))\geq\min(v(d'),v(e'))$$

and

$$v(d)=v(f)+v(d'),v(e)=v(f)+v(e').$$

Note that if $v(d'),v(e')>0$, then $0\geq\min(v(d'),v(e'))>0$, which gives a contradiction. Thus one of $v(d'),v(e')$ has to be zero, and we get the result. □

Corollary 3.2.6. The partial order $\big(\operatorname{ADiv}_{\mathbb{Q}}(F),\leq\big)$ is a lattice, where $\wedge$ is the minimum.

Proof. This follows from (2) in Lemma 3.2.5 and Lemma 2.2.9. $\square$

Note that maximum in $\mathrm{ADiv}_{\mathbb{Q}}(F)$ is calculated by $\overline{\mathscr{D}} \vee \overline{\mathscr{E}} = -\big((-\overline{\mathscr{D}}) \wedge (-\overline{\mathscr{E}})\big)$.

Definition 3.2.7. We denote by $\mathrm{LDiv}_{\mathbb{Q}}(F)$ the $\mathbb{Q}$-vector space generated by $\mathbb{Q}$-vector and lattice operations from principal arithmetic $\mathbb{Z}$-divisors $\widehat{\mathrm{div}}(f)$, for $f \in F$. We call these *lattice divisors* on F. For an F-model $\mathscr{X}$ we denote by $\mathrm{LDiv}_{\mathbb{Q}}(\mathscr{X})$ the inverse image of lattice divisors on F via the natural map $\mathrm{ADiv}_{\mathbb{Q}}(\mathscr{X}) \to \mathrm{ADiv}_{\mathbb{Q}}(F)$. We denote by $\mathrm{LDiv}^0_{\mathbb{Q}}(F), \mathrm{LDiv}^0_{\mathbb{Q}}(\mathscr{X})$ the subspaces where the corresponding divisor is 0.

Definition 3.2.8. Let l be a linear functional on a (real or rational) vector subspace of $\mathrm{ADiv}_{\mathbb{R}}(F)$ (resp. $\mathbb{Q}$ or $\mathbb{R}$-linear). We call l a *globally valued field functional* (abbreviated GVF functional), if it satisfies the following properties:

- it maps principal $\mathbb{R}$-divisors of C^0-type in its domain to 0,
- it is non-negative on effective arithmetic $\mathbb{R}$-divisors of C^0-type in its domain.

Moreover, we call l *normalised*, if its domain contains a pullback of a non-principal arithmetic $\mathbb{R}$-divisor of C^0-type $\overline{\mathscr{D}}$ on $\mathrm{Spec}(\mathbb{Z})$ and l sends it to $\widehat{\deg}(\overline{\mathscr{D}})$. Here we mean the arithmetic degree from Definition 2.2.5, for the arithmetic variety $\mathrm{Spec}(\mathbb{Z})$. If the domain contains the zero divisor $\overline{\mathscr{N}}$ with its Green's function constantly equal 2, then it is equivalent to $l(\overline{\mathscr{N}}) = 1$.

Lemma 3.2.9. Every GVF functional

$$\mathrm{ADiv}_{\mathbb{Q}}(F) \to \mathbb{R}$$

extends uniquely to a GVF functional

$$\mathrm{ADiv}_{\mathbb{R}}(F) \to \mathbb{R}.$$

Proof. By Lemma 2.2.11 we get the uniqueness. By [25, Proposition 2.4.2] the natural map $\mathrm{ADiv}_{\mathbb{Q}}(F) \otimes \mathbb{R} \to \mathrm{ADiv}_{\mathbb{R}}(F)$ is surjective. Existence follows from the fact that the real span of the image of the effective cone of $\mathrm{ADiv}_{\mathbb{Q}}(F)$ spans the effective cone in $\mathrm{ADiv}_{\mathbb{R}}(F)$. $\square$

Proposition 3.2.10. Let $\mathscr{X}$ be an F-model. Then $\mathrm{LDiv}^0_{\mathbb{Q}}(\mathscr{X})$ is a dense subset of $\mathrm{ADiv}^0_{\mathbb{R}}(\mathscr{X})$ with respect to the supremum norm.

Proof. Identify $\mathrm{ADiv}^0_{\mathbb{R}}(\mathscr{X})$ with continuous real-valued functions on $\mathscr{X}(\mathbb{C})/F_\infty$. By the Stone–Weierstrass theorem (max-closed), it is enough to check the following:

- a constant non-zero function is in $\mathrm{LDiv}^0_{\mathbb{Q}}(\mathscr{X})$;
- if $g \in \mathrm{LDiv}^0_{\mathbb{Q}}(\mathscr{X})$, then $ag \in \mathrm{LDiv}^0_{\mathbb{Q}}(\mathscr{X})$ for $a \in \mathbb{Q}$;
- if $g, h \in \mathrm{LDiv}^0_{\mathbb{Q}}(\mathscr{X})$, then $g+h, \max(g,h) \in \mathrm{LDiv}^0_{\mathbb{Q}}(\mathscr{X})$;
- the algebra $\mathrm{LDiv}^0_{\mathbb{Q}}(\mathscr{X})$ separates points of $\mathscr{X}(\mathbb{C})/F_\infty$.

For a constant non-zero function we can take $\widehat{\mathrm{div}}(2) \wedge \widehat{\mathrm{div}}(1)$. The second and third points are automatic. Now we prove the last point. Fix an embedding $\mathscr{X} \subset \mathbb{P}^n_{\mathbb{Z}}$ and assume that $\mathscr{X}$ is not contained in any hyperplane. Pick

$$x, y \in \mathscr{X}(\mathbb{C}) \subset \mathbb{P}^n_{\mathbb{Z}}(\mathbb{C}) = \{[z_0 : \cdots : z_n] | (\exists i = 0, \ldots, n)(z_i \neq 0)\}$$

which have different conjugation classes. Assume that $x_0 = y_0 = 1$ (possibly after a change of coordinates of $\mathbb{P}^n_{\mathbb{Z}}$). Let i, i' be indexes among $1, \ldots, n$ such that $x_i \neq y_i$ and $\overline{x_{i'}} \neq y_{i'}$. Note that we can without loss of generality assume that $i' = i$.

Indeed, we consider two cases. First, if x_i or $x_{i'}$ is real, then $x_i = \overline{x_i} \neq y_i$ or $x_{i'} = \overline{x_{i'}} \neq y_{i'}$ respectively, and we win in both cases. Second, if $x_i, x_{i'} \in \mathbb{C} \setminus \mathbb{R}$ and $\overline{x_i} = y_i, x_{i'} = y_{i'}$, then we can change the coordinates of $\mathbb{P}^n_{\mathbb{Z}}$ so that the new i'th coordinate is a sum of the i'th and i''th coordinate, and other coordinates stay the same. The new i'th coordinates of x and y become $x_i + x_{i'}, \overline{x_i} + x_{i'}$ respectively, and they have different conjugation classes as $x_i, x_{i'} \in \mathbb{C} \setminus \mathbb{R}$.

Let $X_0, \ldots, X_n$ be the natural projective coordinates on $\mathbb{P}^n_{\mathbb{Z}}$ and denote by f_j the restriction of $\frac{X_j}{X_0}$ to $\mathscr{X}$, for each $j = 0, \ldots, n$. This makes sense as $\mathscr{X}$ is not contained in any hyperplane. Let $\overline{\mathscr{E}}$ be the lattice divisor on F defined by the formula

$$\overline{\mathscr{E}} := \widehat{\operatorname{div}}(1) \wedge \widehat{\operatorname{div}}(f_1) \wedge \cdots \wedge \widehat{\operatorname{div}}(f_{i-1}) \wedge \widehat{\operatorname{div}}(f_{i+1}) \wedge \cdots \wedge \widehat{\operatorname{div}}(f_n).$$

Pick natural numbers a, b such that the following hold:

(1) $\left|\frac{x_i+a}{x_i+b}\right| \neq \left|\frac{y_i+a}{y_i+b}\right|$;

(2) for $c = a$ or $c = b$ we have

$$|x_i + c| > 1, |x_1|, \ldots, |x_n|;$$
$$|y_i + c| > 1, |y_1|, \ldots, |y_n|.$$

Such exists by Lemma 3.2.12 with $N \geq 2\max_{i=0,\ldots,n} |x_i|$. Consider the following lattice divisor on F

$$\overline{\mathscr{D}} := -(\overline{\mathscr{E}} \wedge \widehat{\operatorname{div}}(f_i + a)) + (\overline{\mathscr{E}} \wedge \widehat{\operatorname{div}}(f_i + b)).$$

Let $\mathscr{X}'$ be an F-model over $\mathscr{X}$ with $\overline{\mathscr{D}} \in \operatorname{LDiv}_{\mathbb{Q}}(\mathscr{X}')$.

Claim 3.2.11. Let v be a non-Archimedean valuation on F. Then $\beta(v, \overline{\mathscr{D}}) = 0$.

Proof of the claim. By Lemma 3.2.5 we calculate:

$$\begin{aligned}\beta(v, \overline{\mathscr{D}}) &= \beta(v, \overline{\mathscr{E}} \wedge \widehat{\operatorname{div}}(f_i + b)) - \beta(v, \overline{\mathscr{E}} \wedge \widehat{\operatorname{div}}(f_i + a)) \\ &= \min(\{v(f_j) : j \neq i\} \cup \{v(f_i + b), 0\}) - \min(\{v(f_j) : j \neq i\} \cup \{v(f_i + a), 0\}),\end{aligned}$$

where j varies form 1 to n with $f_0 = 1$. Note that

$$v(f_i + a) = v((f_i + b) + (a - b)) \geq \min(v(f_i + b), v(a - b)) \geq \min(v(f_i + b), 0)$$

and symmetrically $v(f_i + b) \geq \min(v(f_i + a), 0)$ so $\min(v(f_i + a), 0) = \min(v(f_i + b), 0)$, which finishes the proof. ☐

In particular, the $\mathbb{R}$-Weil divisor associated to $\mathscr{D}$ is trivial, hence $\mathscr{D} = 0$ as $\mathscr{X}'$ is normal. Thus $\overline{\mathscr{D}} = (0, g)$ for some C^0-type function g on $\mathscr{X}'(\mathbb{C})$. Consider the function H defined by the formula

$$-\min(\{-\log|z_j| : j \neq i\} \cup \{-\log|z_i + az_0|\}) + \min(\{-\log|z_j| : j \neq i\} \cup \{-\log|z_i + bz_0|\}),$$

for $[z_0 : \cdots : z_n] \in \mathbb{P}^n_{\mathbb{Z}}(\mathbb{C})$. Note that it is a C^0-type function on $\mathbb{P}^n_{\mathbb{Z}}(\mathbb{C})$ (it does not depend on the representative of a point) and it restricts to a C^0-type function $h := H|_{\mathscr{X}(\mathbb{C})}$ on $\mathscr{X}(\mathbb{C})$. By Lemma 3.2.5 and Definition 2.2.2, for points $p \in \mathscr{X}'(\mathbb{C})$ factoring through the inclusion $\operatorname{Spec}(F) \to \mathscr{X}'$ (note that those are the same for $\mathscr{X}$ and $\mathscr{X}'$), we have

$$\begin{aligned}g(p) &= -\min(\{-\log|f_j|(p) : j \neq i\} \cup \{-\log|f_i + a|(p)\}) \\ &\quad + \min(\{-\log|f_j|(p) : j \neq i\} \cup \{-\log|f_i + b|(p)\}) \\ &= -\min(\{-\log|z_j|(p) : j \neq i\} \cup \{-\log|z_i + az_0|(p)\})\end{aligned}$$

$$+\min(\{-\log|z_j|(p) : j \neq i\} \cup \{-\log|z_i + bz_0|(p)\})$$
$$= H(p) = h(p),$$

where the second inequality holds as $f_i = \frac{X_i}{X_0}|_{\mathscr{X}}$ and for p as above $z_0(p) \neq 0$. Since g, h are continuous, g must be a pullback of h to $\mathscr{X}'(\mathbb{C})$, as those agree on a dense subset. Thus $(0, h) \in \mathrm{LDiv}_{\mathbb{Q}}(\mathscr{X})$. Moreover, by the choice of a, b we get that

$$h(x) = \log\left|\frac{x_i + a}{x_i + b}\right| \neq \log\left|\frac{y_i + a}{y_i + b}\right| = h(y).$$

This finishes the proof. $\square$

Lemma 3.2.12. Let $x \neq y \in \mathbb{C}$ be such that $x \neq y, \overline{x} \neq y$. Let N be a natural number. Then there are natural numbers $a, b > N$ such that

$$\left|\frac{x+a}{x+b}\right| \neq \left|\frac{y+a}{y+b}\right|$$

Proof. Note that for a fixed real number $r > 0$ and natural $a \neq b$ the equation $\left|\frac{x+a}{x+b}\right| = r$ defines a circle of Apollonius with the center in $\mathbb{R}$. There are infinitely many Apollonius circles defined by having fixed ratios of distances to some two different natural numbers a, b. Thus, if the conclusion of the lemma was not true, points x and y would lie on the intersection of (at least) two different circles centered on $\mathbb{R}$. This is only possible if $x = y$ or $\overline{x} = y$. $\square$

Lemma 3.2.13. Let $\mathscr{D}$ be a $\mathbb{Q}$-Cartier divisor on an F-model $\mathscr{X}$. Then there exists a $\mathscr{D}$-Green's function g on $\mathscr{X}$ such that $(\mathscr{D}, g) \in \mathrm{LDiv}_{\mathbb{Q}}(\mathscr{X})$.

Proof. By presenting $\mathscr{D}$ as a rational number times a difference of two very ample Cartier divisors, we can reduce to the case where $\mathscr{D}$ is very ample. In particular it is globally generated. Pick a basis of rational functions $f_1, \ldots, f_n \in H^0(\mathscr{X}, \mathscr{D})$.

Claim 3.2.14. $\mathscr{D} + \bigwedge_{i=1}^n \mathrm{div}(f_i) = 0$.

Proof of the claim. Recall that for a Cartier divisor $\mathscr{D}$ its sections are

$$H^0(\mathscr{X}, \mathscr{D}) = \{0\} \cup \{f \in \kappa(\mathscr{X})^\times \mid \mathscr{D} + \mathrm{div}(f) \geq 0\}.$$

Note that $\{\mathscr{D} + \mathrm{div}(f) \mid f \in H^0(\mathscr{X}, \mathscr{D})\}$ is the set of effective Cartier divisors linearly equivalent to $\mathscr{D}$. The fact that $\mathscr{O}(\mathscr{D})$ is globally generated implies that for every $p \in \mathscr{X}$ there is $\mathscr{E} \in \{\mathscr{D} + \mathrm{div}(f_i) \mid i = 1, \ldots, n\}$ such that $p \notin \mathscr{E}$. Thus the ideal defining $\mathscr{E}$ is (1) at p. Since p was arbitrary, we get that the sum of ideals of effective Cartier divisors $\mathscr{E}_i = \mathscr{D} + \mathrm{div}(f_i)$ is (1). By the definition this means that

$$0 = \bigwedge_{i=1,\ldots,n} (\mathscr{D} + \mathrm{div}(f_i)) = \mathscr{D} + \bigwedge_{i=1,\ldots,n} \mathrm{div}(f_i),$$

which finishes the proof of the claim. $\square$

Thus, we can put $\overline{\mathscr{D}} := -\bigwedge_{i=1}^n \widehat{\mathrm{div}}(f_i)$, so that $g = \log\max_{i=1,\ldots,n}|f_i|^2$ is a desired $\mathscr{D}$-Green's function on $\mathscr{X}$. $\square$

Proposition 3.2.15. Every GVF functional

$$\mathrm{LDiv}_{\mathbb{Q}}(F) \to \mathbb{R}$$

extends uniquely to a GVF functional

$$\mathrm{ADiv}_{\mathbb{R}}(F) \to \mathbb{R}.$$

Proof. Let $l : \mathrm{LDiv}_{\mathbb{Q}}(F) \to \mathbb{R}$ be a GVF functional. By Lemma 3.2.9 it is enough to show that l has a unique extension to $\mathrm{ADiv}_{\mathbb{Q}}(F)$. Pick $\overline{\mathscr{D}} \in \mathrm{ADiv}_{\mathbb{Q}}(F)$ and let $\mathscr{X}$ be a generically smooth, normal, arithmetic variety with $\kappa(\mathscr{X}) \simeq F$ such that $\overline{\mathscr{D}} = (\mathscr{D}, g) \in \mathrm{ADiv}_{\mathbb{Q}}(\mathscr{X})$. By Lemma 3.2.13 (possibly replacing $\mathscr{X}$ by an F-model over it) there is a $\mathscr{D}$-Green's function g' on $\mathscr{X}$ with $(\mathscr{D}, g') \in \mathrm{LDiv}_{\mathbb{Q}}(\mathscr{X})$. The value of l on $(\mathscr{D}, g')$ is determined, so it is enough to define (and show uniqueness of) the value of the extension at $(0, g-g') \in \mathrm{ADiv}_{\mathbb{Q}}(\mathscr{X})$. Let h be the continuous extension of $g - g'$ to $\mathscr{X}(\mathbb{C})$. By Proposition 3.2.10 there is a pair of Green's functions for the 0 divisor g_1, g_2 on $\mathscr{X}$ such that:

- $g_1 \leq h \leq g_2$,
- $0 \leq g_2 - g_1 < \varepsilon$,
- $(0, g_1), (0, g_2) \in \mathrm{LDiv}_{\mathbb{Q}}(\mathscr{X})$.

To find these Green's functions, take g_1 approximating $h - \frac{1}{4}\varepsilon$ up to $\frac{1}{4}\varepsilon$ and similarly with g_2. Hence, any GVF extension of l assigns a value between $l\big((0, g_1)\big)$ and $l\big((0, g_2)\big)$ to $(0, h)$. Moreover,

$$|l\big((0, g_2)\big) - l\big((0, g_1)\big)| = |l\big((0, g_2 - g_1)\big)| < \varepsilon |l\big((0, 1)\big)|.$$

Take a sequence $\varepsilon_n \to 0$ and for each ε_n pick $(g_1)_n, (g_2)_n$ as above, such that $(g_1)_n$ is increasing and $(g_2)_n$ is decreasing (one can do it since $\mathrm{LDiv}_{\mathbb{Q}}(F)$ is closed under maximum and minimum). For any GVF extension of l, we must have $l((0, h)) = \lim_n l((0, (g_i)_n)$ for $i = 1, 2$.

This proves that if an extension exists, it has to be given by the above construction. The existence follows from Theorem 3.4.1. □

3.3. **Existential closedness of the algebraic numbers.** In this section we introduce globally valued fields and compare them to GVF functionals. For different equivalent definitions and the relationship to adelic curves [15], see [5].

Definition 3.3.1. Let F be a field. A *global height* on F is a collection of functions $h : \mathbb{P}^n(F) \to \mathbb{R}_{\geq 0}$, for every natural n, satisfying the following axioms.

Height of one:		$h(1:1) = 0$
Invariance:	$\forall x \in \mathbb{P}^n(F)$, $\forall \sigma \in \mathrm{Sym}_{n+1}$,	$h(\sigma x) = h(x)$
Additivity:	$\forall x \in \mathbb{P}^n(F)$, $\forall y \in \mathbb{P}^m(F)$,	$h(x \otimes y) = h(x) + h(y)$
Monotonicity:	$\forall x \in \mathbb{P}^n(F)$, $\forall y \in \mathbb{P}^m(F)$,	$h(x) \leq h(x : y)$
Triangle inequality:	$\forall x, y \in F^n, x + y \neq 0$,	$h(x + y) \leq h(x : y) + e$

for some real number $e \geq 0$ which is called the Archimedean error, and $(x_i)_i \otimes (y_j)_j = (x_i y_j)_{i,j}$ is the Segre embedding. A *globally valued field* is a field equipped with a global height. We denote $\mathrm{ht}(x) := h(x : 1)$.

In the notation the dependence on n is implicit. For any globally valued field F its Archimedean error can be taken to be $\mathrm{ht}(2)$. Note that if we multiply a global height by a scalar $\lambda \geq 0$ we get a new one. We call a global height (or a GVF structure) *trivial*, if h is everywhere zero.

Example 3.3.2 (Number fields)**.** Let K be a number field and consider a tuple $x_0, \ldots, x_n \in K$ defining a point $x = [x_0 : \ldots : x_n] \in \mathbb{P}^n(K)$. Let P_K be the set of places on K. The Weil logarithmic height of x is defined as

$$h_{\mathrm{Weil}}(x) = \frac{1}{[K : \mathbb{Q}]} \sum_{v \in P_K} [K_v : \mathbb{Q}_v] \log \max_{i=0,\ldots,n} |x_i|_v,$$

where $|\cdot|_v$ is the norm corresponding to a place v on K normalised so that either $|p|_v = p^{-1}$ for some prime p, or $|\cdot|_v$ extends the Euclidean absolute value on $\mathbb{Q}$.

It is easy to check that Weil logarithmic height satisfies the axioms from Definition 3.3.1. Moreover, for $x \in \mathbb{P}^n(\overline{\mathbb{Q}})$, the value $h(x)$ is independent of a number field K for which $x \in \mathbb{P}^n(K)$. Thus, Weil logarithmic height defines a GVF structure on $\overline{\mathbb{Q}}$. It is Galois-invariant, and in fact unique up to scalar multiplication, by Dirichlet's unit theorem and finiteness of class groups of number fields (see for example [5, Lemma 10.2]).

In [5] the authors describe a partially ordered group "$\mathrm{LDiv}_{\mathbb{Q}}(F)$" (denoted by $\mathrm{LDiv}_{\mathbb{Q}}(F)$ in loc. cit. but we write "-" to differentiate from Definition 3.2.3) associated to a field F, such that global heights on F correspond to global positive functionals on "$\mathrm{LDiv}_{\mathbb{Q}}(F)$". In fact, "$\mathrm{LDiv}_{\mathbb{Q}}(F)$" is a *group lattice*, which means that it is a lattice with respect to the ambient order. In [5] $\mathrm{ULat}_{\mathbb{Q}}(F)$ is the divisible group lattice envelope of the monoid of finite subsets of $F^{\times}$. Next, a pairing between $\mathrm{ULat}_{\mathbb{Q}}(F)$ and norms on F is defined, and "$\mathrm{LDiv}_{\mathbb{Q}}(F)$" is the quotient of $\mathrm{ULat}_{\mathbb{Q}}(F)$ by the kernel of this pairing. The clash of notation is not an accident.

Proposition 3.3.3. The group lattice $\mathrm{LDiv}_{\mathbb{Q}}(F)$ is isomorphic to the one introduced in [5, Definition 5.1].

Proof. From the universal property, there is a surjective group lattice morphism from $\mathrm{ULat}_{\mathbb{Q}}(F)$ to the Arakelov lattice divisors $\mathrm{LDiv}_{\mathbb{Q}}(F)$. By Lemma 3.2.5 and induction, the pairing $\beta(v, \overline{\mathscr{D}})$ coincides with the pairing ev from [5, Definition 4.7], for $\overline{\mathscr{D}} \in \mathrm{LDiv}_{\mathbb{Q}}(F)$. Furthermore, by Lemma 2.2.9 together with [5, Lemma 6.12], the kernel of the map $\mathrm{ULat}_{\mathbb{Q}}(F) \to \mathrm{LDiv}_{\mathbb{Q}}(F)$ coincides with [5, Definition 5.1]. $\square$

As a corollary, we get Theorem C.

Corollary 3.3.4. Taking a GVF functional $l : \mathrm{ADiv}_{\mathbb{R}}(F) \to \mathbb{R}$ to a global height on F defined by

$$h(x_0 : \ldots : x_n) = l\Big(-\bigwedge_{i=0}^{n} \widehat{\mathrm{div}}(x_i)\Big)$$

defines a bijection

$$\big\{\text{GVF functionals } \mathrm{ADiv}_{\mathbb{R}}(F) \to \mathbb{R}\big\} \longrightarrow \big\{\text{Global heights on } F\big\}.$$

Proof. This follows from [5, Theorem 1.2], Proposition 3.3.3 and Proposition 3.2.15. $\square$

Definition 3.3.5. Let F be a GVF. It is *existentially closed*, if for any GVF extension $F \subset K$ and for any finite tuple $\overline{a} = (a_1, \ldots, a_q)$ in K the following condition holds: Assume that m, n, l are natural numbers and

$$f_1, \ldots, f_m \in F[x_1, \ldots, x_q]^l, g_1, \ldots, g_n, p \in F[x_1, \ldots, x_q]$$

satisfy

$$\text{(A)} \qquad \begin{cases} h(1 : f_{i,1}(\overline{a}) : \ldots : f_{i,l}(\overline{a})) = r_i \text{ for } i = 1, \ldots, m; \\ g_1(\overline{a}) = \ldots = g_n(\overline{a}) = 0; \\ p(\overline{a}) \neq 0; \end{cases}$$

where h is the global height on K and we write $f_i = (f_{i,1}, \ldots, f_{i,l})$ for $i = 1, \ldots, m$. Then for any $\varepsilon > 0$ there is a tuple $\overline{a}' = (a_1', \ldots, a_q')$ in F such that

$$\text{(A')} \qquad \begin{cases} |h(1 : f_{i,1}(\overline{a}') : \ldots : f_{i,l}(\overline{a}')) - r_i| < \varepsilon \text{ for } i = 1, \ldots, m; \\ g_1(\overline{a}') = \ldots = g_n(\overline{a}') = 0; \\ p(\overline{a}') \neq 0. \end{cases}$$

At this point we can prove Theorem A.

Theorem 3.3.6. $\overline{\mathbb{Q}}$ with the Weil logarithmic height is existentially closed.

Proof. We start with the following reduction.

Claim 3.3.7. To prove existential closedness of $\overline{\mathbb{Q}}$ it suffices to prove the following statement:

($\heartsuit$) For every finitely generated GVF extension $\mathbb{Q} \subset K$ generated by a tuple $\overline{a}$, if $\overline{a}$ satisfies Equations (A) in Definition 3.3.5 (with $F = \mathbb{Q}$), then for any $\varepsilon > 0$ there exists a tuple $\overline{a}'$ in $\overline{\mathbb{Q}}$ satisfying Equations (A').

Proof of the claim. Let $\overline{\mathbb{Q}} \subset K$ be a GVF extension and let $\overline{a} = (a_1, \ldots, a_q)$ be a tuple in K. Assume that it satisfies Equations (A) from Definition 3.3.5 (with $F = \overline{\mathbb{Q}}$). Let b be an element in $\overline{\mathbb{Q}}$ such that f_i, g_j, p are defined using parameters from $\mathbb{Q}(b)$ for all i, j as in Definition 3.3.5. Fix $\varepsilon > 0$. We use the condition $(*)$ for the finitely generated extension $\mathbb{Q} \subset \mathbb{Q}(\overline{a}, b)$ with the Equations (A) together with $g_{n+1}(x_{q+1})$ being the minimal polynomial of b over $\mathbb{Q}$. We get a tuple $(\overline{a}', b')$ satisfying Equations (A'), but with b' replacing b in all functions f_i, g_j, p, and with b' having the same minimal polynomial as b. Let σ be an automorphism of $\overline{\mathbb{Q}}$ taking b' to b. Since the GVF structure $\overline{\mathbb{Q}}$ is Galois-invariant (see Example 3.3.2) the tuple $\sigma(\overline{a}')$ satisfies Equations (A'), which finishes the proof of the claim. $\square$

We prove ($\heartsuit$) for a finitely generated GVF extension $\mathbb{Q} \subset K = \mathbb{Q}(\overline{a})$ with $\overline{a} = (a_1, \ldots, a_q)$. By uniqueness of finite GVF extensions of $\mathbb{Q}$, we can assume that K is not algebraic over $\mathbb{Q}$. Let X be the smallest subvariety of $\mathbb{A}^q$ containing $\overline{a}$ (so the ideal that defines it is the locus of $\overline{a}$ over $\mathbb{Q}$) and write $F_i : X \to \mathbb{A}^l$ for the composition of $X \subset \mathbb{A}^q$ with the morphism $\mathbb{A}^q \to \mathbb{A}^l$ corresponding to f_i, for $i = 1, \ldots, m$. Pick a K-model $\mathscr{X}$ such that there are maps $h_i : \mathscr{X} \to \mathbb{P}^l_{\mathbb{Z}}$ and an open $U \subset X$ making the following diagram commute

$$\begin{tikzcd} U \arrow[r, hook] \arrow[dr, "r"'] & X \arrow[r, "F_i"] & \mathbb{A}^l_{\mathbb{Q}} \arrow[d, "j"] \\ & \mathscr{X} \arrow[r, "d_i"] & \mathbb{P}^l_{\mathbb{Z}} \end{tikzcd}$$

where $j : \mathbb{A}^l_{\mathbb{Q}} \to \mathbb{P}^l_{\mathbb{Z}}$ is the map taking $(z_1, \ldots, z_l)$ to $[1 : z_1 : \cdots : z_l]$. By making U smaller we can without loss of generality assume that $p(x_1, \ldots, x_q) \neq 0$ on U (with respect to natural coordinates on $U \subset \mathbb{A}^q$). Define an arithmetic $\mathbb{R}$-divisor of C^0-type on $\mathbb{P}^l_{\mathbb{Z}}$ by $-\bigwedge_{c=0}^{l} \widehat{\operatorname{div}}(\frac{z_c}{z_0})$ or equivalently

$$\overline{\mathscr{W}} = \left(\{z_0 = 0\}, -2 \log \frac{|z_0|}{\min(|z_0|, \ldots, |z_l|)} \right),$$

and put $\overline{\mathscr{D}}_i := d_i^* \overline{\mathscr{W}}$, for $i = 1, \ldots, m$. Take any $\varepsilon > 0$. Define a GVF functional $l : \operatorname{ADiv}_{\mathbb{Q}}(\mathscr{X}) \to \mathbb{R}$ by composing $\operatorname{ADiv}_{\mathbb{Q}}(\mathscr{X}) \to \operatorname{ADiv}_{\mathbb{Q}}(K)$ with the GVF functional corresponding to the global height on K. By Theorem 3.1.4 there is $x \in U(\overline{\mathbb{Q}})$ such that

$$|h_{\overline{\mathscr{D}}_i}(x) - l(\overline{\mathscr{D}}_i)| < \varepsilon$$

for $i = 1, \dots, m$. Let $\overline{a}' = (a'_1, \dots, a'_q)$ be the coordinates of x inside $U(\overline{\mathbb{Q}}) \subset \mathbb{A}^q(\overline{\mathbb{Q}})$. Looking at the diagram and using Lemma 2.2.7 we get that for all i,

$$h_{\overline{\mathscr{D}}_i}(x) = h_{\overline{\mathscr{W}}}(d_i(x)) = h_{\mathrm{Weil}}(1 : f_{i,1}(\overline{a}') : \dots : f_{i,l}(\overline{a}')).$$

The second equality uses the fact that $h_{\overline{\mathscr{W}}}$ coincides with the Weil logarithmic height on $\mathbb{P}^l(\overline{\mathbb{Q}})$. To deal with $l(\overline{\mathscr{D}}_i)$ we first prove the following

Claim 3.3.8. For $i = 1, \dots, m$ we have:

$$\overline{\mathscr{D}}_i = -\Big(\bigwedge_{c=1}^{l} \widehat{\mathrm{div}}(f_{i,c}(\overline{a})) \wedge 0\Big),$$

where on the right hand side we skip the terms where $f_{i,c}(\overline{a}) = 0$.

Proof of the claim. On the level of Cartier divisors, this follows from Claim 3.2.14, as rational functions $\frac{z_c}{z_0}$ pullback via d_i to $f_{i,c}(\overline{a})$ for $c = 1, \dots, l$, and to 1 for $c = 0$. On the level of Green's functions it follows from the definition of a pullback. □

By Corollary 3.3.4 it follows that for $i = 1, \dots, m$ we have:

$$l(\overline{\mathscr{D}}_i) = h(1 : f_{i,1}(\overline{a}) : \dots : f_{i,l}(\overline{a})).$$

This finishes the proof, because $\overline{a}'$ lies in X which is the smallest subvariety of $\mathbb{A}^q$ that contains $\overline{a}$. □

3.4. **Essential infimum and GVF functionals.** For this section, let F be a finitely generated extension of $\mathbb{Q}$ and let $\mathscr{X}$ be an F-model. Assume that the transcendence degree of F over $\mathbb{Q}$ is d (in particular $\mathscr{X}$ is of dimension $d+1$).

Theorem 3.4.1. (M.Riesz extension theorem) Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{Q}$ and work in the category of $\mathbb{K}$ vector spaces. Let E be a vector space, $W \subset E$ be a subspace, and let $K \subset E$ be a convex cone. Assume that a linear functional $l_0 : W \to \mathbb{R}$ non-negative on $W \cap K$ is given. If the condition

$$E \subset K + W$$

is satisfied, then there exists an extension $l : E \to \mathbb{R}$ of ϕ, non-negative on K. Moreover, l is unique, if for each $e \in E$ and $\varepsilon > 0$, there are $w, w' \in W$ with $w - e, e - w' \in K$ and $l_0(w - w') < \varepsilon$.

Corollary 3.4.2. Let V be a real vector subspace of $\mathrm{ADiv}_{\mathbb{R}}(F)$ containing a big arithmetic $\mathbb{R}$-divisor $\overline{\mathscr{D}}$ of C^0-type. Assume that we are given a GVF functional

$$l_0 : V \to \mathbb{R}.$$

Then there exists an extension $l_0 \subset l : \mathrm{ADiv}_{\mathbb{R}}(F) \to \mathbb{R}$ to a GVF functional l.

Proof. Let W be the image of V under the projection $\pi : \mathrm{ADiv}_{\mathbb{R}}(F) \to \mathrm{APic}_{\mathbb{R}}(F)$. Note that l_0 factors through W as it is a GVF functional. Use Theorem 3.4.1 (the existence part) with $E = \mathrm{APic}_{\mathbb{R}}(F)$ and K being the image of the cone of effective arithmetic $\mathbb{R}$-divisors of C^0-type under π. Take a class of an arithmetic $\mathbb{R}$-divisor of

C^0-type $\overline{\mathscr{E}}$ on an F-model $\mathscr{X}$. Note that by Theorem 2.3.3, for a natural number n we have (possibly by passing to a different F-model)

$$\widehat{\mathrm{vol}}(n\overline{\mathscr{D}}+\overline{\mathscr{E}}) = n^{d+1}\widehat{\mathrm{vol}}(\overline{\mathscr{D}}+\frac{1}{n}\overline{\mathscr{E}}),$$

and $\lim_n \widehat{\mathrm{vol}}(\overline{\mathscr{D}}+\frac{1}{n}\overline{\mathscr{E}}) = \widehat{\mathrm{vol}}(\overline{\mathscr{D}}) > 0$, so for n big enough $\overline{\mathscr{F}} := n\overline{\mathscr{D}}+\overline{\mathscr{E}}$ is big. In particular its π image is in K. Note that $\overline{\mathscr{E}} = \overline{\mathscr{F}} + (-n\overline{\mathscr{D}})$, and by the fact that $\pi(\overline{\mathscr{E}}) \in E$ was arbitrary, we get an extension function $\mathrm{APic}_{\mathbb{R}}(F) \to \mathbb{R}$ non-negative on classes of effective arithmetic $\mathbb{R}$-divisors of C^0-type. By composing it with π we are done. □

Construction 3.4.3. Let $\overline{\mathscr{B}}$ be a big arithmetic $\mathbb{R}$-divisor of C^0-type on an F-model $\mathscr{X}'$. We define a linear functional $l_{\overline{\mathscr{B}}} : \mathrm{ADiv}_{\mathbb{R}}(F) \to \mathbb{R}$ by the following formula.

$$\begin{aligned} l_{\overline{\mathscr{B}}} &: \mathrm{ADiv}_{\mathbb{R}}(F) \to \mathbb{R} \\ \overline{\mathscr{E}} &\mapsto \langle(\phi^*\overline{\mathscr{B}})^d\rangle \cdot \overline{\mathscr{E}}, \end{aligned}$$

where $\overline{\mathscr{E}}$ is an arithmetic $\mathbb{R}$-divisor of C^0-type on $\phi : \mathscr{X}'' \to \mathscr{X}'$ and we use notation from Definition 2.3.2. By the projection formula in arithmetic geometry (see e.g. [23, Lemma 2.3]), $l_{\overline{\mathscr{B}}}$ is well defined. Moreover, by Theorem 2.3.3 we have

$$\langle(\phi^*\overline{\mathscr{B}})^d\rangle \cdot \overline{\mathscr{E}} = \frac{1}{d+1}D_{\overline{\mathscr{B}}}\widehat{\mathrm{vol}}(\overline{\mathscr{E}}) = \lim_{t\to 0^+}\frac{\widehat{\mathrm{vol}}(\phi^*\overline{\mathscr{B}}+t\overline{\mathscr{E}}) - \widehat{\mathrm{vol}}(\phi^*\overline{\mathscr{B}})}{(d+1)t}.$$

But for effective $\overline{\mathscr{E}}$ and $t > 0$ we have $\widehat{\mathrm{vol}}(\phi^*\overline{\mathscr{B}}+t\overline{\mathscr{E}}) - \widehat{\mathrm{vol}}(\phi^*\overline{\mathscr{B}}) \geq 0$. Thus $l_{\overline{\mathscr{B}}}(\overline{\mathscr{E}}) \geq 0$ and so $l_{\overline{\mathscr{B}}}$ is in fact a GVF functional. Note that if $\overline{\mathscr{B}}$ is nef, then $l_{\overline{\mathscr{B}}}(\overline{\mathscr{E}}) = (\phi^*\overline{\mathscr{B}})^d \cdot \overline{\mathscr{E}}$.

Lemma 3.4.4. Let $\overline{\mathscr{D}}$ be an arithmetic $\mathbb{R}$-divisor of C^0-type on $\mathscr{X}$. Then $\overline{\mathscr{D}}$ is pseudo-effective if and only if the following holds:

For all (normalised) GVF functionals $l : \mathrm{ADiv}_{\mathbb{R}}(F) \to \mathbb{R}$ we have $l(\overline{\mathscr{D}}) \geq 0$.

Moreover, $\overline{\mathscr{D}}$ is principal if and only if it satisfies the following:

For all (normalised) GVF functionals $l : \mathrm{ADiv}_{\mathbb{R}}(F) \to \mathbb{R}$ we have $l(\overline{\mathscr{D}}) = 0$.

Proof. We start with the first characterisation. Assume that $\overline{\mathscr{D}}$ is pseudo-effective and let $\overline{\mathscr{B}}$ be a big arithmetic $\mathbb{R}$-divisor of C^0-type. Then for every $\delta > 0$ we have $\widehat{\mathrm{vol}}(\overline{\mathscr{D}}+\delta\overline{\mathscr{B}}) > 0$, so if l is a GVF functional on $\mathrm{ADiv}_{\mathbb{R}}(F)$ we have

$$l(\overline{\mathscr{D}}) = \lim_{\delta\to 0^+} l(\overline{\mathscr{D}}+\delta\overline{\mathscr{B}}) \geq 0.$$

On the other hand, if $\overline{\mathscr{D}}$ is not pseudo-effective, by Theorem 2.3.1 there is an F-model $\pi : \mathscr{X}' \to \mathscr{X}$ with a nef arithmetic $\mathbb{R}$-divisor $\overline{\mathscr{H}}$ such that $\overline{\mathscr{H}}^d \cdot \pi^*\overline{\mathscr{D}} < 0$. By Construction 3.4.3, $l = l_{\overline{\mathscr{H}}}$ is a GVF functional, and it satisfies $l(\overline{\mathscr{D}}) < 0$ which finishes the proof. Now we argue that it is enough to look at normalised GVF functionals. Let $\overline{\mathscr{N}}$ be the zero divisor with Green's function 2. Assume that the first condition holds for normalised GVF functionals on $\mathrm{ADiv}_{\mathbb{R}}(F)$. Take any GVF functional $l : \mathrm{ADiv}_{\mathbb{R}}(F) \to \mathbb{R}$. If $l(\overline{\mathscr{N}}) \neq 0$, we can rescale it (using a positive scalar, as $\overline{\mathscr{N}}$ is effective) to be normalised, so $l(\overline{\mathscr{D}}) \geq 0$. If $l(\overline{\mathscr{N}}) = 0$, we can take any normalised GVF functional l' on $\mathrm{ADiv}_{\mathbb{R}}(F)$ and if we had $l(\overline{\mathscr{D}}) < 0$, then for n big enough $(l' + nl)(\overline{\mathscr{D}}) < 0$, which finishes this part of the proof.

To prove the other characterisation, it is enough to prove that if all (normalised) GVF functionals vanish on $\overline{\mathscr{D}}$, then $\overline{\mathscr{D}}$ is principal. Note that by the first part of the lemma we get that $\overline{\mathscr{D}}$ is pseudo-effective. Now the result follows from Theorem 2.3.1 and the proof of the first characterisation. □

For completeness, we reprove Theorem 4.6 from [27] in the setting of GVF functionals, as follows.

Theorem 3.4.5. Let $\overline{\mathscr{D}}$ be an arithmetic $\mathbb{R}$-divisor of C^0-type on $\mathscr{X}$. Then

$$\zeta(\overline{\mathscr{D}}) \leq \inf\{l(\overline{\mathscr{D}}) : l \text{ is a normalised GVF functional on } \mathrm{ADiv}_{\mathbb{R}}(F)\}.$$

Moreover, if $\mathscr{D}_{\mathbb{Q}}$ is big, this is an equality and infimum can be replaced by minimum.

Proof. For the first part, assume that there is a normalised GVF functional $l : \mathrm{ADiv}_{\mathbb{R}}(F) \to \mathbb{R}$ such that $l(\overline{\mathscr{D}}) = r$. Fix a closed strict subscheme $\mathscr{Y} \subset \mathscr{X}$ and $\varepsilon > 0$. By Theorem 3.1.4 there is a closed point $x \in X = \mathscr{X}_{\mathbb{Q}}$ such that $x \notin \mathscr{Y}$ and $|h_{\overline{\mathscr{D}}}(x) - l(\overline{\mathscr{D}})| < \varepsilon$. We can use the theorem because l is normalised. Since $\mathscr{Y}$ was arbitrary, we get

$$\zeta(\overline{\mathscr{D}}) = \sup_{\mathscr{Y} \subset \mathscr{X}} \inf_{x \in \mathscr{X} \setminus \mathscr{Y}(\overline{\mathbb{Q}})} h_{\overline{\mathscr{D}}}(x) \leq r + \varepsilon.$$

As l and ε were arbitrary, we get the desired inequality. For the other part, by Lemma 2.4.3 (using bigness of $\mathscr{D}_{\mathbb{Q}}$) we have

$$\zeta(\overline{\mathscr{D}}) \geq \sup\{r : \overline{\mathscr{D}} - r \cdot \overline{\mathscr{N}} \text{ is pseudo-effective}\},$$

for $\overline{\mathscr{N}}$ being the zero divisor with Green's function 2. By Lemma 3.4.4 we know that for any fixed $r \in \mathbb{R}$, the arithmetic $\mathbb{R}$-divisor $\overline{\mathscr{D}} - r \cdot \overline{\mathscr{N}}$ is pseudo-effective if and only if for all GVF functionals $l : \mathrm{ADiv}_{\mathbb{R}}(F) \to \mathbb{R}$ we have $l(\overline{\mathscr{D}} - r \cdot \overline{\mathscr{N}}) \geq 0$. Note that for a normalised GVF functional $l(\overline{\mathscr{D}} - r \cdot \overline{\mathscr{N}}) = l(\overline{\mathscr{D}}) - r$, so this expression is ≥ 0 if and only if $l(\overline{\mathscr{D}}) \geq r$. By Lemma 3.4.4 we get that

$$\zeta(\overline{\mathscr{D}}) \geq \sup\{r : \forall l,\ l(\overline{\mathscr{D}}) \geq r\},$$

where we quantify over all normalised GVF functionals l on $\mathrm{ADiv}_{\mathbb{R}}(F)$. This gives the desired equality with infimum in the case where $\mathscr{D}_{\mathbb{Q}}$ is big. Consider the vector space V generated by $\overline{\mathscr{N}}$ and $\overline{\mathscr{D}}$ in $\mathrm{ADiv}_{\mathbb{R}}(F)$. Next, define l on V by putting $l(\overline{\mathscr{N}}) := 1, l(\overline{\mathscr{D}}) := \zeta(\overline{\mathscr{D}})$. By Corollary 3.4.6 we get that l is a GVF functional, and by Corollary 3.4.2 it has an extension to $\mathrm{ADiv}_{\mathbb{R}}(F)$, which proves that infimum can be replaced by maximum. □

As a corollary one recovers [27, Theorem 1.4] which generalises [2, Corollary 4.2] as it removes the semi-positivity assumption.

Corollary 3.4.6. Let $\overline{\mathscr{D}}$ be an arithmetic $\mathbb{R}$-divisor of C^0-type on $\mathscr{X}$ with $\mathscr{D}_{\mathbb{Q}}$ big and let $\overline{\mathscr{N}}$ be the zero divisor with Green's function 2. Then

$$\zeta(\overline{\mathscr{D}}) = \sup\{r : \overline{\mathscr{D}} - r \cdot \overline{\mathscr{N}} \text{ is pseudo-effective}\}.$$

Proof. Note that by Lemma 2.4.3

$$\zeta(\overline{\mathscr{D}}) \geq r \iff \zeta(\overline{\mathscr{D}} - r\overline{\mathscr{N}}) \geq 0,$$

which by Theorem 3.4.5 is equivalent to the inequality $\inf\{l(\overline{\mathscr{D}}) - r\} \geq 0$ where the infimum is taken over all normalised GVF functionals on $\mathrm{ADiv}_{\mathbb{R}}(F)$. By Lemma 3.4.4 this is equivalent to $\overline{\mathscr{D}} - r\overline{\mathscr{N}}$ being pseudo-effective, which finishes the proof. □

Construction 3.4.7. Let $\overline{\mathscr{D}} \in \mathrm{ADiv}_{\mathbb{R}}(F)$ be big (recall that then also $\mathscr{D}_{\mathbb{Q}}$ is big). Let

$$\varphi(\overline{\mathscr{E}}) = \frac{\widehat{\mathrm{vol}}(\overline{\mathscr{E}})}{(d+1)\,\mathrm{vol}(\mathscr{E}_{\mathbb{Q}})}$$

for $\overline{\mathscr{E}} \in \mathrm{ADiv}_{\mathbb{R}}(F)$ with big $\mathscr{E}_{\mathbb{Q}}$. Note that it is well defined since both $\widehat{\mathrm{vol}}$ and vol are invariant under birational pullbacks. By differentiability of $\widehat{\mathrm{vol}}$ from Theorem 2.3.3 and differentiability of vol proved in [9, Theorem A] we get a linear functional

$$D_{\overline{\mathscr{D}}}\varphi : \mathrm{ADiv}_{\mathbb{R}}(F) \to \mathbb{R}$$

which is the Gateaux derivative of φ calculated on an appropriate F-model. This linear functional is normalised and zero on principal arithmetic $\mathbb{R}$-divisors of C^0-type, but is not necessarily a GVF functional.

Theorem 3.4.8. Let $\overline{\mathscr{D}}$ be a big arithmetic $\mathbb{R}$-divisor of C^0-type on $\mathscr{X}$. Consider the statements:

(1) $\varphi(\overline{\mathscr{D}}) = \zeta(\overline{\mathscr{D}})$;
(2) $D_{\overline{\mathscr{D}}}\varphi$ is a GVF functional;
(3) the infimum

$$\zeta(\overline{\mathscr{D}}) = \inf\{l(\overline{\mathscr{D}}) : l \text{ is a normalised GVF functional on } \mathrm{ADiv}_{\mathbb{R}}(F)\}$$

is achieved at the unique normalised GVF functional;
(4) ζ is Gateaux differentiable at $\overline{\mathscr{D}}$ in every direction.

Then (1) $\iff$ (2) $\implies$ (3) $\iff$ (4).

Proof. Let $\overline{\mathscr{E}} \in \mathrm{ADiv}_{\mathbb{R}}(F)$. Note that for small real t the arithmetic $\mathbb{R}$-divisor of C^0-type $\overline{\mathscr{D}} + t\overline{\mathscr{E}}$ is big. Let l_0 be a normalised GVF functional on $\mathrm{ADiv}_{\mathbb{R}}(F)$ with $l_0(\overline{\mathscr{D}}) = \zeta(\overline{\mathscr{D}})$. Such exists by Theorem 3.4.5. For real $t > 0$ small enough we then have

$$(\spadesuit) \qquad \frac{\zeta(\overline{\mathscr{D}} + t\overline{\mathscr{E}}) - \zeta(\overline{\mathscr{D}})}{t} \leq \frac{l_0(\overline{\mathscr{D}} + t\overline{\mathscr{E}}) - l_0(\overline{\mathscr{D}})}{t} = l_0(\overline{\mathscr{E}}).$$

To get the above inequality we use Theorem 3.4.5. Now we pass to proving equivalence of the conditions in the theorem.

(1) $\Rightarrow$ (2) : Assuming $\varphi(\overline{\mathscr{D}}) = \zeta(\overline{\mathscr{D}})$, by Theorem 2.4.4 we get

$$\frac{\varphi(\overline{\mathscr{D}} + t\overline{\mathscr{E}}) - \varphi(\overline{\mathscr{D}})}{t} \leq \frac{\zeta(\overline{\mathscr{D}} + t\overline{\mathscr{E}}) - \zeta(\overline{\mathscr{D}})}{t} \leq l_0(\overline{\mathscr{E}}).$$

Taking $t \to 0^+$ we get that $D_{\overline{\mathscr{D}}}\varphi \leq l_0$, but since they are linear (in particular odd) this implies that $D_{\overline{\mathscr{D}}}\varphi = l_0$ is a GVF functional.

(2) $\Rightarrow$ (1) : From Theorem 3.4.5 we then get that $\zeta(\overline{\mathscr{D}}) \leq D_{\overline{\mathscr{D}}}\varphi(\overline{\mathscr{D}}) = \varphi(\overline{\mathscr{D}})$ which together with Theorem 2.4.4 gives the equality.

(1) $\Rightarrow$ (4) : From the proof of (1) $\Rightarrow$ (2) we get that

$$l_0(\overline{\mathscr{E}}) = D_{\overline{\mathscr{D}}}\varphi(\overline{\mathscr{E}}) \leq \lim_{t\to 0^+} \frac{\zeta(\overline{\mathscr{D}} + t\overline{\mathscr{E}}) - \zeta(\overline{\mathscr{D}})}{t} \leq l_0(\overline{\mathscr{E}}).$$

For the negative t replace $\overline{\mathscr{E}}$ by $-\overline{\mathscr{E}}$.

(4) $\Rightarrow$ (3) : By taking the limit $t \to 0$ in Equation ($\spadesuit$) we get

$$D_{\overline{\mathscr{D}}}\zeta(\overline{\mathscr{E}}) \leq l_0(\overline{\mathscr{E}}).$$

Since both maps are linear, we get $D_{\overline{\mathscr{D}}}\zeta = l_0$, hence the uniqueness of l_0 follows.

(3) $\Rightarrow$ (4) : Let V be the real vector subspace generated by $\overline{\mathscr{N}}, \overline{\mathscr{D}}, \overline{\mathscr{E}}$ in $\mathrm{ADiv}_{\mathbb{R}}(F)$, where $\overline{\mathscr{N}}$ is the zero divisor with Green's function 2. If the classes of $\overline{\mathscr{N}}, \overline{\mathscr{D}}, \overline{\mathscr{E}}$ are linearly dependent in $\mathrm{APic}_{\mathbb{R}}(F)$, then differentiability of ζ at $\overline{\mathscr{D}}$ in direction $\overline{\mathscr{E}}$ follows from Lemma 2.4.2 and Lemma 2.4.3. Assume that those classes are linearly independent. Let V^+ be the (Euclidean) closure of the big cone in V and pick a hyperplane P in V containing $\overline{\mathscr{N}}, \overline{\mathscr{D}}$, such that $Q := P \cap V^+$ is a compact convex set. This is possible as V^+ does not contain a line. By assumption and Theorem 3.4.1 we know that l_0 restricted to P is the unique affine function l on P having the following properties:

- $l(\overline{\mathscr{N}}) = 1$,
- $l(\overline{\mathscr{D}}) = \zeta(\overline{\mathscr{D}})$,
- l is non-negative on Q.

Consider the function $f(t) = \zeta(\overline{\mathscr{D}} + t\overline{\mathscr{E}})$ on a neighbourhood of 0. It is concave by Lemma 2.4.2. We now show that if f is not differentiable at zero, then we can find a different affine function l having the listed properties. By moving the hypersurface P and rescaling $\overline{\mathscr{N}}, \overline{\mathscr{D}}, \overline{\mathscr{E}}$ we can without loss of generality assume that $\overline{\mathscr{N}}, \overline{\mathscr{D}}, \overline{\mathscr{E}} \in P$. In other words

$$P = \{a\overline{\mathscr{N}} + b\overline{\mathscr{D}} + c\overline{\mathscr{E}} : a + b + c = 1\}.$$

Fix coordinates on P so that $\overline{\mathscr{N}}$ is the zero and the basis vectors are $e_1 = \overline{\mathscr{D}} - \overline{\mathscr{N}}, e_2 = \overline{\mathscr{E}} - \overline{\mathscr{N}}$. Consider the function

$$g(\varepsilon) := \sup\{r : \overline{\mathscr{D}} + \varepsilon e_2 + r e_1 \text{ is pseudo-effective}\}.$$

There is a positive t such that $g : (-t, t) \to \mathbb{R}$ is a concave function, as Q is a compact convex set. Note that

$$\overline{\mathscr{D}} + \varepsilon e_2 + r e_1 = (1 + r)\overline{\mathscr{D}} + \varepsilon\overline{\mathscr{E}} - (r + \varepsilon)\overline{\mathscr{N}}.$$

By Corollary 3.4.6 we know that $\overline{\mathscr{D}} + \varepsilon e_2 + r e_1$ is pseudo-effective if and only if $\zeta(\overline{\mathscr{D}} + \varepsilon e_2 + r e_1) \geq 0$, which by the above calculation and Lemma 2.4.3 is equivalent to

$$\zeta(\overline{\mathscr{D}} + \varepsilon e_2 + r e_1) = \zeta((1 + r)\overline{\mathscr{D}} + \varepsilon\overline{\mathscr{E}}) - (r + \varepsilon)$$
$$= (1 + r)\zeta\Big(\overline{\mathscr{D}} + \frac{\varepsilon}{1 + r}\overline{\mathscr{E}}\Big) - (r + \varepsilon) = (1 + r)f\Big(\frac{\varepsilon}{1 + r}\Big) - (r + \varepsilon) \geq 0.$$

The supremum of r where this quantity is non-negative is achieved when we have the equality with zero, so we get the functional equation

$$(1 + g(\varepsilon))f\Big(\frac{\varepsilon}{1 + g(\varepsilon)}\Big) - (g(\varepsilon) + \varepsilon) = 0.$$

Thus

$$f\Big(\frac{\varepsilon}{1 + g(\varepsilon)}\Big) = \frac{g(\varepsilon) + \varepsilon}{1 + g(\varepsilon)}.$$

Note that $g(0)$ is strictly positive as $\widehat{\mathrm{vol}}(\overline{\mathscr{D}}) > 0$. If g was differentiable at 0, then f also would be, as

$$\frac{f\big(\frac{\varepsilon}{1+g(\varepsilon)}\big) - f(0)}{\frac{\varepsilon}{1+g(\varepsilon)}} = \frac{\frac{g(\varepsilon)+\varepsilon}{1+g(\varepsilon)} - \frac{g(0)}{1+g(0)}}{\frac{\varepsilon}{1+g(\varepsilon)}}.$$

Thus g is not differentiable at 0. This means that there exist two different lines X_1, X_2 on P such that $X_i \cap Q = \{\overline{\mathscr{D}} + g(0)e_1\}$ for $i = 1, 2$. Consider affine functions l_1, l_2 on P determined by the following conditions

- l_i vanishes on X_i,
- $l_i(\overline{\mathscr{N}}) = 1$,
- $l_i(\overline{\mathscr{D}}) = \zeta(\overline{\mathscr{D}})$,

for $i = 1, 2$. Note that such functions exists by Corollary 3.4.6. Moreover, they are non-negative on Q, since they are affine, their zero sets are supporting lines for Q, and they are positive on a point $\overline{\mathscr{N}}$ in Q. This gives a non-uniqueness of l_0, which finishes the proof. □

Question 3.4.9. What is the set of all arithmetic big $\mathbb{R}$-divisors of C^0-type $\overline{\mathscr{D}}$ with $D_{\overline{\mathscr{D}}}\zeta$ Gateaux differentiable?

Before the next lemma, note that if $x \in \mathscr{X}_{\mathbb{Q}}$ is a sufficiently general closed point, then for $\overline{\mathscr{E}} \in \mathrm{ADiv}_{\mathbb{R}}(F)$ one can make sense out of $h_{\overline{\mathscr{E}}}(x)$. Indeed, if $\overline{\mathscr{E}} \in \mathrm{ADiv}_{\mathbb{R}}(\mathscr{X}')$ for some F-model $\phi : \mathscr{X}' \to \mathscr{X}$, then if x is in the open subset where ϕ is an isomorphism, we can define $h_{\overline{\mathscr{E}}}(x) := h_{\overline{\mathscr{E}}}(\phi^{-1}(x))$.

The below lemma follows the lines of [11, Lemma (8.2)] or [14, Section 5] and proves that differentiability of ζ is enough for equidistribution (over all places).

Lemma 3.4.10. Let $\overline{\mathscr{D}}$ be a big arithmetic $\mathbb{R}$-divisor of C^0-type on $\mathscr{X}$. Assume that $(x_n)_{n\in\mathbb{N}}$ is a sequence of closed points in the generic fiber of $\mathscr{X}$ which converges to the generic point. If ζ is Gateaux differentiable at $\overline{\mathscr{D}}$ in directions $\overline{\mathscr{E}}, -\overline{\mathscr{E}} \in \mathrm{ADiv}_{\mathbb{R}}(F)$ and

$$\lim_n h_{\overline{\mathscr{D}}}(x_n) = \zeta(\overline{\mathscr{D}}),$$

then

$$\lim_n h_{\overline{\mathscr{E}}}(x_n) = D_{\overline{\mathscr{D}}}\zeta(\overline{\mathscr{E}}).$$

Proof. Let $t > 0$ be a real number. We calculate

$$\liminf_{n\to\infty} h_{\overline{\mathscr{E}}}(x_n) = \liminf_{n\to\infty} \frac{h_{\overline{\mathscr{D}}+t\overline{\mathscr{E}}}(x_n) - h_{\overline{\mathscr{D}}}(x_n)}{t}.$$

Note that $\liminf_{n\to\infty} h_{\overline{\mathscr{D}}+t\overline{\mathscr{E}}}(x_n) \geq \zeta(\overline{\mathscr{D}} + t\overline{\mathscr{E}})$ and $\liminf_{n\to\infty} h_{\overline{\mathscr{D}}}(x_n) = \zeta(\overline{\mathscr{D}})$. Thus we get

$$\liminf_{n\to\infty} \frac{h_{\overline{\mathscr{D}}+t\overline{\mathscr{E}}}(x_n) - h_{\overline{\mathscr{D}}}(x_n)}{t} \geq \frac{\zeta(\overline{\mathscr{D}} + t\overline{\mathscr{E}}) - \zeta(\overline{\mathscr{D}})}{t}.$$

Taking the limit $t \to 0^+$ we have

$$\liminf_{n\to\infty} h_{\overline{\mathscr{E}}}(x_n) \geq D_{\overline{\mathscr{D}}}\zeta(\overline{\mathscr{E}}).$$

Replacing $\overline{\mathscr{E}}$ with $-\overline{\mathscr{E}}$ we get

$$\limsup_{n\to\infty} h_{\overline{\mathscr{E}}}(x_n) \leq D_{\overline{\mathscr{D}}}\zeta(\overline{\mathscr{E}}).$$

Together those inequalities give $\lim_n h_{\overline{\mathscr{E}}}(x_n) = D_{\overline{\mathscr{D}}}\zeta(\overline{\mathscr{E}})$ which finishes the proof. □

Definition 3.4.11. Let $\overline{\mathscr{D}}, \overline{\mathscr{E}}$ be arithmetic $\mathbb{R}$-divisors of C^0-type. Say that $\overline{\mathscr{D}}$ *satisfies logarithmic equidistribution in direction* $\overline{\mathscr{E}}$, if for any generic sequence $x_n \in \mathscr{X}(\overline{\mathbb{Q}})$ with $h_{\overline{\mathscr{D}}}(x_n) \to \zeta(\overline{\mathscr{D}})$, the limit $\lim_n h_{\overline{\mathscr{E}}}(x_n)$ exists and does not depend on $(x_n)_{n\in\mathbb{N}}$.

Corollary 3.4.12. Let $\overline{\mathscr{D}}$ be a big arithmetic $\mathbb{R}$-divisor of C^0-type on $\mathscr{X}$ and $\overline{\mathscr{E}} \in \mathrm{ADiv}_{\mathbb{Q}}(\mathscr{X})$. Then $\overline{\mathscr{D}}$ satisfies logarithmic equidistribution in direction $\overline{\mathscr{E}}$, if and only if ζ is Gateaux differentiable at $\overline{\mathscr{D}}$ in direction $\overline{\mathscr{E}}$.

Proof. Lemma 3.4.10 proves the implication from right to left. On the other hand, if $\overline{\mathscr{D}}$ does not satisfy logarithmic distribution in direction $\overline{\mathscr{E}}$, then there are generic sequences $(x_n)_{n\in\mathbb{N}}, (x'_n)_{n\in\mathbb{N}}$ on an $\mathscr{X}$, such that

$$\lim_n h_{\overline{\mathscr{D}}}(x_n) = \lim_n h_{\overline{\mathscr{D}}}(x'_n) = \zeta(\overline{\mathscr{D}})$$

and

$$\lim_n h_{\overline{\mathscr{E}}}(x_n) \neq \lim_n h_{\overline{\mathscr{E}}}(x'_n).$$

In particular, there are two different GVF functionals on the linear span of $\overline{\mathscr{D}}, \overline{\mathscr{E}}$ in $\mathrm{ADiv}_{\mathbb{R}}(F)$ so Theorem 3.4.1 and Theorem 3.4.8 finish the proof. $\square$

The bigness condition was relaxed in [3]. Moreover, the authors made progress towards Question 3.4.9 by finding new criteria (e.g. [3, Theorem 6.1]) for differentiability of the essential infimum ζ.

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