MODEL THEORY

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Contents

1. Introduction

Model theory is a part of logic where one studies classes of models of various sets of axioms. The fundamental questions of the area are: how to classify/construct models, what configurations can appear in a model, and how are different theories (sets of axioms) connected to each other. The standard methods of model theory are set up in the framework of first-order logic, i.e., one can only take quantifiers over elements, not over subsets. In applications it is often a non-trivial task to translate a problem into a first-order equivalent, but when possible it can lead to strong results.

Every theory is expressed in a language, e.g., groups have group multiplication, rings have $+, \cdot$, and vector spaces over a field have $+$ and multiplications by scalars. Equality is assumed to be in the language by default.

Example 1.1. Consider the theory

$$T = \left\{ (\exists x_1, \ldots, x_n) \left(\bigwedge_{i \neq j} x_i \neq x_j \right) : n \in \mathbb{N} \right\}.$$

Models of T are just infinite sets $M \models T$. Up to isomorphism, these are classified by their cardinality. For a formula

$$\varphi(\overline{x}) = \exists y_1 \forall y_2 \ldots \exists y_{2n-1} \forall y_{2n} \bigwedge_i \bigvee_j (\neg) \alpha_{ij} = \beta_{ij},$$

where $\alpha_{ij}, \beta_{ij} \in \{x_a, y_b : a, b\}$, one can show e that $\varphi(M) \subseteq M^{|\overline{x}|}$ is a Boolean combination of weak diagonals.

Of course, some properties of infinite sets depend on the model of ZFC we are working with, but we will treat the cardinal hierarchy as absolute, and rather reduce questions about models to, for example, cardinality questions. Surprisingly, many properties of the class of models of a theory are independent of the model of ZFC we choose. For a discussion on why we are working with a metatheory, see [3, 4].

For example, regardless of the universe of set theory, for the above theory T its spectrum function

$$I(T, \kappa) := |\{M \models T : |M| = \kappa\}/\text{iso.}|$$

is constantly equal to 1. A theory T such that $I(T, \kappa) = 1$ is called *κ-categorical*, and characterisation of such theories (for $\kappa = \aleph_0$ or $\kappa > \aleph_0$) is one of the most fundamental results in model theory (Ryll-Nardzewski's and Morley's theorem, respectively).

The scope of model theory has increased substantially since its origin. The bounds of the area are ultimately defined by what its users can prove. Nonetheless, there are several slogans describing what model theory is, that perhaps reflect the history of the development of the subject:

[Chang and Keisler, 1973] Model theory = Logic + Universal algebra,

[Hodges, 1997] Model theory = Algebraic geometry - Fields,

[Hrushovski] Model theory is the geography of tame mathematics.

These will hopefully become more meaningful during the course-time. The term "geography" in Hrushovski's description may be loosely treated as a reference to Shelah's idea of drawing division lines on the map of all theories: defining properties of theories that reflect the structure of their models. This point of view is exemplified quite literally in the model theoretic "map of the universe" [2].

Each 'geographical' region has a different culture and can lead to different interesting applications. During the course we can venture into various model-theoretic continents including frameworks for:

- geometry of differential equations and differential Galois theory (stability);
- difference equations, pseudo-finite fields (simplicity);
- tame topology and real-algebraic geometry (o-minimality);
- valued fields, tame non-Archimedean geometry (NIP, generic stability).

Depending on the interests of the audience we will choose one or more directions and from time to time we will supplement examples from the corresponding regions.

2. Completeness theorem

The fundamental tool of model theory is Gödel's completeness theorem. It was proven by Kurt Gödel in his PhD thesis in 1929 and reformulated in Leon Henkin's PhD thesis in 1947. The theorem says that every first-order statement that is true in every model of some theory T admits a proof from the axioms of T. To state the completeness theorem formally we need to set the stage.

Definition 2.1. A *language* L consists of collections of *function symbols* $\{f_i\}_{i \in I}$, *relation symbols* $\{P_j\}_{j \in J}$, and *constant symbols* $\{c_k\}_{k \in K}$. Furthermore, each function and relation symbol is equipped with its *arity*, which is a natural number n_i for $i \in I$, m_j for $j \in J$.

Fix a language L for the rest of this section.

Definition 2.2. An L-structure is a non-empty set M together with *interpretations* of:

- functions $f_i^M : M^{n_i} \to M$ for $i \in I$,

- relations $P_j^M \subseteq M^{m_j}$ for $j \in J$,
- constants $c_k^M \in M$ for $k \in K$.

Definition 2.3. Fix an infinite set of variables $\{v\}_{v\in V}$. The *L-terms* (in variables $\{v\}_{v\in V}$) are elements of the smallest set that contains the variables, constant symbols, and whenever $i \in I$, $t_1, \dots, t_{n_i}$ are terms, then so is $f_i(t_1, \dots, t_{n_i})$.

We define the set of *L-formulas* (in variables $\{v\}_{v\in V}$) inductively as follows:

- An *atomic formula* is either of the form $t_1 = t_2$, where t_1, t_2 are terms in variables $\{v\}_{v\in V}$, or of the form $P_j(t_1, \dots, t_{m_j})$ for some $j \in J$ and terms $t_1, \dots, t_{m_j}$.
- If φ is a formula, then so is $\neg\varphi$.
- If φ, ψ are formulas, then so are $\varphi \wedge \psi, \varphi \vee \psi$.
- If φ is a formula and $v \in V$, then so are $(\exists v)(\varphi)$ and $(\forall v)(\varphi)$.

It would be enough to use e.g. $\wedge, \neg, \exists$ as logical symbols, and define other ones as combinations of those, for example $(\forall x)(\varphi) := \neg(\exists x)(\neg\varphi)$. Similarly, we use the conventions $(\varphi \vee \psi) := \neg(\neg\varphi \wedge \neg\psi)$, $(\varphi \to \psi) := (\neg\varphi \vee \psi)$ and $(\varphi \leftrightarrow \psi) := (\varphi \to \psi) \wedge (\psi \to \varphi)$. Also, we write $\bigwedge_{i=1}^{l} \varphi_i, \bigvee_{i=1}^{l} \varphi_i$ for the inductively defined iterations of $\wedge, \vee$ respectively. We often skip parentheses, L and variables in the names, if those are clear from the context. We fix an infinite set of variables $\{v\}_{v\in V}$ for the rest of this section.

For every term t and formula φ we inductively define the set of variables $FV(t)$, $FV(\varphi)$ occurring freely in t, φ respectively, so that:

(i) $FV(v) = \{v\}, FV(c_k) = \varnothing, FV(f_i(t_1, \dots, t_{n_i})) = \bigcup_{a=1}^{n_i} FV(t_a)$.
(ii) $FV(\neg\varphi) = FV(\varphi), FV(\varphi \wedge \psi) = FV(\varphi) \cup FV(\psi)$.
(iii) $FV(\exists v\ \varphi) = FV(\varphi) \setminus \{v\}$.

We often denote variables by $x_1, \dots, x_n$. We write $t(x_1, \dots, x_n)$ or $\varphi(x_1, \dots, x_n)$ to mean that the free variables of t or φ respectively, are among $x_1, \dots, x_n$. We use overlines as tuples of variables $\overline{x} = (x_1, \dots, x_n)$.

Definition 2.4. A formula without free variables is called a *sentence*. A set of sentences is called an *L-theory*.

If M is an L-structure and $t(x_1, \dots, x_n)$ is a term, we inductively define the interpretation of t as a function $t^M : M^n \to M$, or as an element of M if $FV(t) = \varnothing$. If t is a single variable or a constant c_k, t^M is the appropriate projection function $M^n \to M$ or $c_k^M \in M$ respectively. If $t = f_i(t_1, \dots, t_{n_i})$, then t^M is defined by applying f_i^M to the values of $t_1^M, \dots, t_{n_i}^M$ on the appropriate tuple of elements of M. The precise definition of this step is left as an exercise e.

Definition 2.5 (Tarski's definition of truth)**.** Let M be an L-structure. We define a relation $M \models \varphi(a_1, \dots, a_n)$ for formulas $\varphi(x_1, \dots, x_n)$ and (possibly empty) tuples $\overline{a} = (a_1, \dots, a_n) \in M^n$ inductively as follows:

(1) If $\varphi(\overline{x})$ is $t_1(\overline{x}) = t_2(\overline{x})$, then $M \models \varphi(\overline{a})$ if $t_1^M(\overline{a}) = t_2^M(\overline{a})$.
(2) If $\varphi(\overline{x})$ is $P_j(t_1(\overline{x}), \dots, t_{m_j}(\overline{x}))$ then $M \models \varphi(\overline{a})$ if $(t_1^M(\overline{a}), \dots, t_{m_j}^M(\overline{a})) \in P_j^M$.
(3) $M \models \neg\varphi(\overline{a})$ if it is not the case that $M \models \varphi(\overline{a})$ (written $M \not\models \varphi(\overline{a})$).
(4) $M \models (\varphi \wedge \psi)(\overline{a})$ if $M \models \varphi(\overline{a})$ and $M \models \psi(\overline{a})$.
(5) If $\varphi(\overline{x}) = (\exists y)(\psi(\overline{x}, y))$, then $M \models \varphi(\overline{a})$ if there is $b \in M$ such that $M \models \psi(\overline{a}, b)$.

If $M \models \varphi(\overline{a})$, we say that M *models* $\varphi(\overline{a})$. If ψ has free variables $x_1, \dots, x_n$, by convention we define $M \models \psi$ to mean $M \models \overline{\psi}$, where $\overline{\psi} = \forall x_1 \dots \forall x_n\ \psi$ is the *universal closure of* ψ. For an L-theory T we say that an L-structure M is a *model of* T, written $M \models T$, if for every $\varphi \in T$, $M \models \varphi$.

In order to state Gödel's completeness theorem it remains to define the notion of a proof. First, recall that a *propositional formula* is an expression built using propositional symbols $P_1, P_2, \dots$ and Boolean connectives $\neg, \wedge$ (we can also add $\vee, \rightarrow, \leftrightarrow$, or define them similarly as before). We say that a propositional formula is a *tautology*, if under any substitution of $Truth$ and $False$ for P_i's, the resulting term in the Boolean algebra $\{Truth, False\}$ is evaluated as $Truth$. If $\tau(P_1, \dots, P_n)$ is a propositional formula, and $\varphi_1, \dots, \varphi_n$ are formulas, we call $\tau(\varphi_1, \dots, \varphi_n)$ an *instantiation* of τ.

Lemma 2.6. If $\tau(P_1, \dots, P_n)$ is a tautology and $\varphi_1, \dots, \varphi_n$ are formulas, then for all L-structures M we have

$$M \models \tau(\varphi_1, \dots, \varphi_n).$$

Proof. Exercise e. □

Definition 2.7. The *axioms of the classical first-order logic* (in the language L and variables $\{v\}_{v \in V}$) consist of:

A0. Instantiations $\tau(\varphi_1, \dots, \varphi_n)$, for tautologies τ and formulas $\varphi_1, \dots, \varphi_n$.

A1. Formulas $\forall x(\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \forall x\ \psi)$, for all pairs of formulas φ, ψ, such that x is not free in φ.

A2. Formulas $\forall x\ \varphi \rightarrow \varphi(x/t)$, for all formulas φ and terms t such that none of the free occurrences of x in φ is in the range of a quantifier bounding a free variable from t. Here $\varphi(x/t)$ is the result of substitution of every free occurrence of x in φ by t.

A3. Equality axioms:
- $x = x$ for every variable x;
- $x = y \rightarrow t(\dots, x, \dots) = t(\dots, y, \dots)$, for every term t and every pair of variables x, y, where y appears in the same place in t as x was;
- $x = y \rightarrow (\varphi(\dots, x, \dots) \leftrightarrow \varphi(\dots, y, \dots))$, for every formula φ and every pair of variables x, y, where y appears in the same place in φ as x was, and x occurs freely in φ, not in the range of a quantifier bounding y.

To be extremely formal in the above definition we would have to define $\varphi(x/t)$ and conditions like "none of the occurrences of x in φ is in the range of a quantifier bounding a free variable from t" inductively. We leave it as an exercise e to the reader. If $\varphi(\overline{x})$ is a formula and $\overline{c}$ is tuple of constants of the same length, then we write $\varphi(\overline{c})$ for the substitution of respective x_i's by c_i's. Note that the restriction in A2 makes sense, as if $\varphi = (\exists y)(x \neq y)$ and $t = y$, then it should not be an axiom that

$$\forall x \exists y\ x \neq y \rightarrow (\exists y)(y \neq y).$$

Definition 2.8. The *deduction rules of the classical first-order logic* consist of the modus ponens (MP) rule

$$\frac{\varphi, \varphi \rightarrow \psi}{\psi}$$

and the $\forall$-rule

$$\frac{\varphi}{\forall x\ \varphi},$$

for every pair of formulas φ, ψ.

Definition 2.9. We say that a set of formulas X proves a formula ψ, written $X \vdash \psi$, if there is a finite sequence of formulas $\alpha_1, \dots, \alpha_n = \psi$, such that for every $i = 1, \dots, n$ one of the following two conditions holds:

(i) α_i is an axiom of the classical first-order logic or an element of X.
(ii) α_i can be deduced from $\alpha_1, \dots, \alpha_{i-1}$ using one of the deduction rules. More precisely, there are $j, k < i$ such that $\alpha_k = (\alpha_j \to \alpha_i)$, or $\alpha_i = \forall x\ \alpha_k$.

If $X = \varnothing$ in the above definition, we just write $\vdash \psi$ instead of $\varnothing \vdash \psi$.

Example 2.10. Let us show that $\vdash \forall x\ \psi \to \exists x\ \psi$. Let β be the formula we want to prove and let y be a variable not appearing in β. The proof can be constructed as follows:

α_1: $\forall x\ \psi \to \psi(x/y)$, by A2.
α_2: $\forall x\ \neg\psi \to \neg\psi(x/y)$, by A2.
α_3: $\alpha_2 \to (\psi(x/y) \to \neg\forall x \neg\psi)$, by A0.
α_4: $\psi(x/y) \to \neg\forall x\neg\psi$, by MP.
α_5: $\alpha_1 \to (\alpha_4 \to \beta)$, by A0.
α_6: $\alpha_4 \to \beta$, by MP.
α_7: β by MP.

Exercise 2.11 (Deduction theorem)**.** e Let φ be a sentence, ψ be a formula, and X be a set of formulas. Then:

$$X \vdash \varphi \to \psi \iff X \cup \{\varphi\} \vdash \psi.$$

Exercise 2.12. e For any formula φ we have $\varphi \vdash \overline{\varphi}$ and $\overline{\varphi} \vdash \varphi$.

For a formula φ and a theory T, we write $T \models \varphi$, if for every L-structure $M \models T$ we have $M \models \varphi$. If $T = \varnothing$ we abbreviate it as $\models \varphi$.

Theorem 2.13 (Soundness)**.** Let φ be a formula. If $\vdash \varphi$, then $\models \varphi$.

Proof. By induction, left as an exercise. e. □

Note that by the finitary nature of proofs and the deduction lemma, it follows that for a theory T, $T \vdash \psi$ implies that $T \models \psi$.

Theorem 2.14 (Gödel's completeness)**.** Let φ be a formula. If $\models \varphi$, then $\vdash \varphi$.

A set of formulas S is *consistent*, if $S \nvdash (\exists x)(x \neq x)$.

Theorem 2.15 (Completeness after Henkin)**.** Let S be a consistent set of sentences. Then S has a model.

Proof. Let κ be the maximum of cardinalities of the number of variables and size of the language. Consider a new language L' which has the same symbols as L together with κ-many new constant symbols $(d_\lambda)_{\lambda<\kappa}$. Let $(\varphi_i(x))_{i<\kappa}$ be a sequence of L'-formulas with one free variable x. Let $g : \kappa \to \kappa$ be an increasing function such that for every $\lambda < \kappa$, $d_{g(\lambda)}$ does not appear in $\varphi_i(x)$ for $i \leq \lambda$. Now consider the set of L'-sentences

$$S' = S \cup \{\exists x\ \varphi_i(x) \to \varphi_i(d_{g(i)}) : i < \kappa\}.$$

We claim that S' is consistent. Indeed, if it were not, there would be $i_1 < \cdots < i_n$ such that

$$S'' = S \cup \{\exists x\ \varphi_{i_j}(x) \to \varphi_{i_j}(d_{g(i_j)}) : j = 1, \ldots, n-1\}$$

is consistent, and it becomes inconsistent if $\exists x\ \varphi_{i_n}(x) \to \varphi_{i_n}(d_{g(i_n)})$ is added to it. Thus, $S'' \vdash \exists x\ \varphi_{i_n}(x) \wedge \neg\varphi_{i_n}(d_{g(i_n)})$. In particular $S'' \vdash \neg\varphi(d_{g(i_n)})$. By the definition of $g(i_n)$, $d_{g(i_n)}$ does not appear in formulas from S''. Hence, we can rewrite the proof witnessing that $S'' \vdash \neg\varphi_{i_n}(d_{g(i_n)})$, so that we get $S'' \vdash \neg\varphi_{i_n}(x)$. This contradicts the fact that $S'' \vdash \exists x\ \varphi_{i_n}(x)$.

Let T be a maximal consistent L'-theory containing S'. Such an extension exists by the Kuratowski-Zorn lemma. It is easy to see that for every L'-sentence φ, we have $\varphi \in T$ or $\neg\varphi \in T$ (theories with this property are called *complete*). We consider a relation on the set of constants $\{d_\lambda\}_{\lambda<\kappa}$ defined by

$$d_\lambda \sim d_\mu \iff T \vdash d_\lambda = d_\mu.$$

It is an equivalence relation and we define an L-structure M by considering the set of equivalence classes of $\sim$, and by requiring that:

(1) $f^M([d_{i_1}]_\sim, \ldots, [d_{i_n}]_\sim) = [d_j]_\sim \iff T \vdash f(d_{i_1}, \ldots, d_{i_n}) = d_j$ for every function symbol f of arity n in L;
(2) $([d_{i_1}]_\sim, \ldots, [d_{i_m}]_\sim) \in R^M \iff T \vdash R(d_{i_1}, \ldots, d_{i_m})$ for every relation symbol R of arity m in L;
(3) $c^M = [d_j]_\sim \iff T \vdash c = d_j$, for every constant symbol c in L.

We leave the proof of the fact that these are well defined and do not depend on choices for the reader ⓔ. The existence of the d_j in the cases (1), (3) follows from the fact that for every term t without free variables, in any language, $\vdash \exists x\ t = x$, which we leave as an exercise ⓔ. It remains to show that $M \models \varphi$, for $\varphi \in S$. More generally, we will show that

$$M \models \alpha \iff T \vdash \alpha,$$

for all L'-sentences (not formulas), where we interpret d_i^M as $[d_i]_\sim$. We do it by induction on formula complexity of α:

- If α is an atomic L'-formula, then it must be of the form $R(t_1, \ldots, t_m)$ or $t_1 = t_2$, for some terms $t_1, t_2, \ldots, t_m$ and a relation symbol R of arity m. If $t_i^M = [d_{\lambda_i}]_\sim$ for all i, then $T \vdash t_i = d_{\lambda_i}$ for all i, so e.g.
$$M \models \alpha \iff M \models R([d_{\lambda_1}]_\sim, \ldots, [d_{\lambda_m}]_\sim)$$
$$\iff T \vdash R(d_{\lambda_1}, \ldots, d_{\lambda_m}) \iff T \vdash R(t_1, \ldots, t_m).$$
The case of equality of two terms is the same.
- If α is a conjunction then the equivalence is obvious. If $\alpha = \neg\beta$, then
$$M \models \alpha \iff M \not\models \beta \iff T \not\vdash \beta \iff T \vdash \neg\beta,$$
where the last equivalence uses the fact that T is maximal consistent.
- If $\alpha = \exists x\varphi_i(x)$, then we choose a witness $[d_\lambda]_\sim$ and we get
$$M \models \varphi_i([d_\lambda]_\sim) \iff M \models \varphi_i(d_\lambda) \iff T \vdash \varphi_i(d_\lambda) \implies T \vdash \exists x\ \varphi_i(x).$$
The middle equivalence follows from the inductive hypothesis. On the other hand
$$T \vdash \exists x\ \varphi_i(x) \implies T \vdash \varphi_i(d_{g(i)}) \iff M \models \varphi_i(d_{g(i)})$$
$$\iff M \models \varphi_i([d_{g(i)}]_\sim) \implies M \models \exists x\ \varphi_i(x).$$

Here the first equivalence follows from the inductive hypothesis.

This completes the proof, as $S \subseteq T$. $\square$

Henkin's proof of completeness is important as it provides a step-by-step 'construction'. It can be generalised to construct models with some specific non-first-order properties.

Corollary 2.16. ⓔ Let T be a theory and ψ be a sentence. Then

$$T \models \psi \iff T \vdash \psi.$$

Corollary 2.17 (Compactness theorem)**.** ⓔ Let T be a theory. Then T has a model if and only if for every finite $T_0 \subseteq T$, T_0 has a model.

From now on, we identify theories that have the same logical consequences, i.e., when we write $T = T'$, we mean $T \vdash T'$ and $T' \vdash T$, or equivalently that for any L-structure M, $M \models T \iff M \models T'$.

Exercises, sheet 1.

1. Fill the gaps left in the proofs above. Exercises throughout the text are marked with the symbol ⓔ. Read the proof of completeness theorem carefully.
2. Prove that if X is a set of formulas and $X \vdash \varphi(c)$, where c is a constant not appearing in any formula from X, then $X \vdash \forall x\ \varphi(x)$.
3. Show that in any language for any formula φ there is a prenex normal form formula ξ such that $\vdash \varphi \leftrightarrow \xi$.
4. Let $L = (+, \cdot, <, (c_r)_{r\in\mathbb{R}}, \dots)$, where in (...) we include a unary (i.e., of arity one) function symbol for every function $f : \mathbb{R} \to \mathbb{R}$. Consider $\mathbb{R}$ as an L-structure. Show that there is a model M of $T = Th(\mathbb{R}) = \{\sigma : \mathbb{R} \models \sigma\}$ such that in M there exists $\varepsilon > 0$ with $\varepsilon < c_r^M$ for all $r \in \mathbb{R}_{>0}$. Hint: use compactness theorem for some $L' = L \cup \{\epsilon\}$-theory and put $\varepsilon = \epsilon^M$.
5. Note that the model M from the previous exercise contains an isomorphic copy of $\mathbb{R}$ (even as an L-structure). We will identify this copy with $\mathbb{R}$ itself.
6. Let $\langle\mathbb{R}\rangle \subseteq M$ be the convex hull of $\mathbb{R} \subseteq M$. Note that $x \in \langle\mathbb{R}\rangle$ if and only if $-n < x < n$ for some $n \in \mathbb{N}$. Show that for any such $x \in \langle\mathbb{R}\rangle \subseteq M$ there is a unique $r \in \mathbb{R}$ such that for all $m \in \mathbb{N}$ we have $-1/m < x - r < 1/m$. We denote it by $r = \mathrm{st}(x)$.
7. Prove that $\mathrm{st} : \langle\mathbb{R}\rangle \to \mathbb{R}$ has the following properties: $\mathrm{st}(x + y) = \mathrm{st}(x) + \mathrm{st}(y), \mathrm{st}(xy) = \mathrm{st}(x)\,\mathrm{st}(y)$. Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function and $r \in \mathbb{R}$. Show that $\mathrm{st}(g^M(r + \varepsilon)) = g(r)$.
8. Let f be a function symbol corresponding to a differentiable function and let f' be the function symbol corresponding to its derivative. Show that for $r \in \mathbb{R}$ we have
$$\mathrm{st}\left(\frac{f^M(r+\varepsilon) - f^M(r)}{\varepsilon}\right) = f'(r).$$
In particular, show that the term inside st is in $\langle\mathbb{R}\rangle$.
9. Prove the Leibniz rule $(fg)' = f'g + fg'$ using the above interpretation of the derivative.
10. Read the exercises below. Familiarise yourself with their statements. Their solutions are optional.

Fix a countable first-order language L. The set of L-formulas (in countably many variables) is sometimes denoted by $L_{\omega,\omega}$. This is to indicate that when we form L-formulas, we can only consider finite conjunctions $\bigwedge_{i=1}^n \varphi_i$ (with $n < \omega$) and finite quantifications $\exists x_1 \dots \exists x_m \varphi$ (with $m < \omega$). Given infinite cardinals κ, λ with $\lambda \leq \kappa$ and a set V of variables with $|V| = \kappa$ one can consider $L_{\kappa,\lambda}$-preformulas which are defined similarly as $L_{\omega,\omega}$-formulas (so we start with atomic formulas and have negation) but with the following connectives and quantification steps:

- If $\{\varphi_\alpha\}_{\alpha\in A}$ are $L_{\kappa,\lambda}$-preformulas with $|A| < \kappa$ then so are
$$\bigwedge_{\alpha\in A} \varphi_\alpha, \bigvee_{\alpha\in A} \varphi_\alpha.$$
- If φ is a $L_{\kappa,\lambda}$-preformula and $X \subset V$ satisfies $|X| < \lambda$, then so is $\exists X\ \varphi$.

We distinguish among those $L_{\kappa,\lambda}$-formulas as those that have only fewer than λ free variables. In the following exercise we will be interested in $L_{\omega_1,\omega}$. In the proof system for $L_{\omega_1,\omega}$ we add axioms
$$\bigwedge_{\alpha\in A} \varphi_\alpha \to \varphi_{\alpha_0},$$
for any countable collection of formulas $\{\varphi_\alpha\}_{\alpha\in A}$ and $\alpha_0 \in A$, and the deduction rule scheme
$$\frac{\varphi_0, \varphi_1, \dots}{\bigwedge_{i\in\mathbb{N}} \varphi_i},$$
for every sequence of $L_{\omega_1,\omega}$-formulas $\{\varphi_i\}_{i\in\mathbb{N}}$. Here proofs are infinitely branching trees.

Exercise 2.18. ⓔ Imitate Henkin's proof to show that if an $L_{\omega_1,\omega}$-sentence σ is consistent, then it has a model. Conclude with the completeness theorem for infinitary formulas $\models \varphi \iff \vdash \varphi$.

Some systems of axioms use more than one type of primitive object. For example, Euclid's axioms are about points P, lines l, and the incidence relation between them $P \in l$. One can formalise this in the framework of multisorted (first-order) logic, where a language L consists of:

- Sort symbols $\{S_\alpha\}_{\alpha\in A}$.
- Function symbols $\{f_i\}_{i\in I}$, each equipped with its arity, which is a tuple $(\alpha_1, \dots, \alpha_n; \beta)$ for some $n \in \mathbb{N}$ and $\alpha_1, \dots, \alpha_n, \beta \in A$.
- Predicate symbols $\{P_j\}_{j\in J}$, each equipped with its arity which is a tuple $(\alpha_1, \dots, \alpha_m)$ for some $m \in \mathbb{N}$ and $\alpha_1, \dots, \alpha_m \in A$.
- Constant symbols $\{c_k\}_{k\in K}$, each equipped with a sort of index $\alpha = \alpha(k) \in A$.

In that case, an L-structure M is a collection of non-empty sets S_α^M together with interpretations of other symbols, where if f is a function symbol with arity $(\alpha_1, \dots, \alpha_n; \beta)$, then its interpretation is a function
$$f^M : S_{\alpha_1}^M \times \dots \times S_{\alpha_n}^M \to S_\beta^M.$$
Similarly, if P is a predicate symbol with arity $(\alpha_1, \dots, \alpha_m)$, then its interpretation is a subset
$$P^M \subseteq S_{\alpha_1}^M \times \dots \times S_{\alpha_m}^M.$$
If c is a constant of sort S, then M is equipped with its interpretation $c^M \in S^M$. When defining L-formulas, we need to consider variables decorated by sort symbols so each sort has a different set of variables associated to it.

Exercise 2.19. ⓔ Convince yourself that Henkin's proof of completeness works for multisorted structures.

Exercise 2.20. ⓔ Look at Hilbert's axioms of Euclidean geometry and define an appropriate multisorted first-order language.

Unless stated otherwise, all further results in the course hold for multisorted structures, however for the ease of notation we will be working with a single-sorted language.

Many axiom systems involve a complete metric and so the real numbers in their definition. For example the axioms of Banach spaces or Hilbert spaces are such. Continuous (first-order) logic is a variant of first-order logic, where one can consider some real-valued predicates and the equality is replaced by the metric. In the following by a *continuity modulus* we mean a continuous increasing function

$$\delta : [0,1] \to [0,1]$$

satisfying $\delta(0) = 0$ and $\delta(\varepsilon) > 0$ for $\varepsilon > 0$.

Definition 2.21. A *continuous language* L consists of:

- sort symbols S each equipped with a metric symbol d_S and with a bound $B_S \in \mathbb{R}_{\geq 0}$;
- constant symbols c with their arity as in the multisorted logic;
- function symbols f with the same arities as in the multisorted logic, each equipped with its continuity modulus δ_f;
- predicate symbols P with their arities same as in the multisorted logic, each equipped with its continuity modulus δ_P and with a bound $B_P \in \mathbb{R}_{\geq 0}$.

An L-structure M consists of interpretations of sorts, constant and function symbols, same as in multisorted logic, however, for each sort S its interpretation S^M is equipped with the metric symbol interpretation d_S^M which makes (S^M, d_S^M) a non-empty complete metric space of diameter at most B_S. Moreover, the function symbols interpretations are supposed to respect the continuity modulus in the following sense: for all real ε with $0 < \varepsilon \leq 1$, for every $a_1, \ldots, a_n, b_1, \ldots, b_n$ in appropriate sorts of M, we have

$$\max_{i=1,\ldots,n} d(a_i, b_i) < \delta_f(\varepsilon) \implies d(f(a_1, \ldots, a_n), f(b_1, \ldots, b_n)) \leq \varepsilon,$$

where we skipped interpretation in M (and the indexes for sorts) in the notation. Furthermore, the interpretations of predicate symbols are functions

$$P^M : S^M_{\alpha_1} \times \cdots \times S^M_{\alpha_m} \to [0, B_P]$$

that are required to respect the continuity modulus in the following sense: for all real $1 \geq \varepsilon > 0$ for every $a_1, \ldots, a_m, b_1, \ldots, b_m$ in appropriate sorts of M we have

$$\max_{i=1,\ldots,m} d(a_i, b_i) < \delta_P(\varepsilon) \implies |P(a_1, \ldots, a_m) - P(b_1, \ldots, b_m)| \leq \varepsilon.$$

When we build formulas in continuous logic, terms are defined similarly as in the discrete case. More complicated expressions are constructed by:

(i) atomic formulas - if t_1, t_2 are terms of the same sort S, then $d_S(t_1, t_2)$ is an atomic formula. If $t_1, \ldots, t_n$ are terms of appropriate sorts, and P is a predicate symbol whose arity has matching sorts, then $P(t_1, \ldots, t_n)$ is an atomic formula.

(ii) connectives - if $\varphi_1, \dots, \varphi_n$ are formulas and $u : \mathbb{R}^n \to \mathbb{R}$ is a continuous function, then $u(\varphi_1, \dots, \varphi_n)$ is a formula.
(iii) quantifiers - if φ is a formula then so are $\sup_x \varphi$ and $\inf_x \varphi$.

In some variants of continuous logic one restricts only to $[0,1]$-valued formulas (all the bounds are equal to 1), and connectives are taken to be only $x \mapsto 1-x, x \mapsto x/2, (x,y) \mapsto \max(x-y, 0)$. If M is an L-structure, $\varphi(\overline{x})$ is a formula with free-variables $\overline{x}$ and $\overline{a} \in M^{\overline{x}}$, then one inductively defines $\varphi(\overline{a})^M \in \mathbb{R}$. A sentence is said to hold in M if it is evaluated to zero. It is possibly to formalise the notion of a proof in continuous logic so that a completeness theorem holds, see [1]. Later, we may prove the following theorem, using ultraproducts.

Theorem 2.22 (Compactness for continuous logic)**.** Let T be a continuous logic theory. Assume that for every $\varphi_1, \dots, \varphi_n \in T$ and every $\varepsilon > 0$ there is an L-structure M such that for all $i = 1, \dots, n$ we have $|\varphi_i^M| \leq \varepsilon$. Then T has a model.

Exercise 2.23. ⓔ Axiomatize the following structures in continuous logic: Banach spaces, Hilbert spaces, C*-algebras. Hint: introduce a sort for each ball of radius $n \in \mathbb{N}$.

3. Löwenheim–Skolem theorem

Fix a first-order language L, so if we refer to formulas or sentences we mean the ones over L. We use letters M, N to denote L-structures.

Definition 3.1. (1) A *homomorphism between M and N* is a function $F : M \to N$ such that for every function symbol f of arity n, predicate symbol P of arity m, and constant symbol c we have

$$\begin{cases} F(f^M(a_1, \dots, a_n)) = f^N(F(a_1), \dots, F(a_n)), \\ P^M(b_1, \dots, b_m) \implies P^N(F(b_1), \dots, F(b_m)), \\ F(c^M) = c^N, \end{cases}$$

for any a_i's and b_j's in M. In particular, L-structures form a category, and an *isomorphism* is a homomorphism that has an inverse homomorphism. We write $M \cong N$ to denote that M and N are isomorphic L-structures.

(2) We say that M *is a substructure of* N, written $M \subseteq N$, if corresponding sets are contained in each other, and interpretations of all symbols in N restrict to interpretations of these symbols in M.
(3) We say that M *is elementarily equivalent to* N, written $M \equiv N$, if for every sentence φ we have $M \models \varphi \iff N \models \varphi$.
(4) We say that M *is an elementary substructure of* N, written $M \prec N$, if M is a substructure of N and for any finite tuple $\overline{a} \subseteq M$ and any formula $\varphi(\overline{x})$ with $|\overline{x}| = |\overline{a}|$ we have

$$M \models \varphi(\overline{a}) \iff N \models \varphi(\overline{a}).$$

(5) A function $\alpha : M \to N$ is *elementary* if for any finite tuple $a_1, \dots, a_n \in M$ and any formula $\varphi(\overline{x})$ with $|\overline{x}| = |\overline{a}|$ we have

$$M \models \varphi(a_1, \dots, a_n) \iff N \models \varphi(\alpha(a_1), \dots, \alpha(a_n)).$$

Theorem 3.2 (Tarski-Vaught test)**.** Let A be a subset of N. Then A is a universe of an elementary substructure of N if and only the following condition $(\star)$ holds:

For every formula $\varphi(x,\overline{y})$ and every $\overline{a}\in A^{|\overline{y}|}$ if $N\models \exists x\ \varphi(x,\overline{a})$, then there is $b\in A$ such that $N\models\varphi(b,\overline{a})$.

Proof. First, we assume that A is a universe of an elementary substructure. Then by the assumption

$$N\models \exists x\ \varphi(x,\overline{a}) \iff A\models \exists x\ \varphi(x,\overline{a}),$$

for $\varphi(x,\overline{y})$ and $\overline{a}$ as in $(\star)$, so there is $b\in A$ such that $A\models\varphi(b,\overline{a})$.

Now assume that $(\star)$ holds. Note that this implies that A is closed under N-interpretations of function symbols, as for a function symbol f of arity n, if $a_1,\ldots,a_n\in A$, then by $(\star)$ we get

$$N\models \exists x\ f(a_1,\ldots,a_n)=x \implies \text{there is } b\in A \text{ such that } f(a_1,\ldots,a_n)=b.$$

Now we show that $A\prec N$ inductively, based on the complexity of the formula. Atomic formulas and connective steps are easy e. Consider the formula $\varphi(\overline{y})=\exists x\ \psi(x,\overline{y})$ and assume that ψ satisfies the inductive hypothesis. Let $\overline{a}\in A^{|\overline{y}|}$. We get

$$N\models \exists x\ \psi(x,\overline{a}) \implies M\models\psi(b,\overline{a}) \text{ for some } b\in A$$

$$\iff A\models\psi(b,\overline{a}) \text{ for some } b\in A \iff A\models \exists x\ \psi(x,\overline{a}),$$

where the first implication follows by $(\star)$ and is actually an equivalence, and the middle equivalence follows from the inductive hypothesis. □

By convention, the cardinality of the language is the cardinality of the set of formulas in countably many variables. Thus, it is maximum of the cardinality of the set of symbols and $\aleph_0$.

Corollary 3.3 (Löwenheim–Skolem theorem)**.** Let M be an infinite L-structure and assume that $|L|=\kappa$. Then:

a) There is an $N\prec M$ with $|N|\leq\kappa$.
b) For any $\lambda\geq\max(\kappa,|M|)$ there is an L-structure N with $|N|=\lambda$ and with an elementary embedding $F:M\to N$.

Proof. a) Let $(\varphi_\alpha(x,\overline{y}))_{\alpha<\kappa}$ be an enumeration of all L-formulas where each formula appears cofinally many times. We define a sequence of subsets $(A_\alpha)_{\alpha<\kappa}$ inductively with $A_0=\varnothing$ and A_β defined so that it contains $\bigcup_{\alpha<\beta}A_\alpha$ and for every $\alpha<\beta$ and every $\overline{a}\subseteq A_\alpha$ of the same length as $\overline{y}$ in $\varphi_\alpha(x,\overline{y})$, if $M\models\exists x\ \varphi_\alpha(x,\overline{a})$, then we choose a witness and add it to A_β. If $N=\bigcup_{\alpha<\kappa}A_\alpha$, then $|N|\leq\kappa$ and it satisfies the Tarski-Vaught test, so $N\prec M$.

b) Let $L'=L\cup\{c_a: a\in M\}\cup\{d_\alpha:\alpha<\lambda\}$ and consider the L'-theory

$$T'=\{\varphi(c_{a_1},\ldots,c_{a_n}):\ \text{if } M\models\varphi(a_1,\ldots,a_n)\}\cup\{d_\alpha\neq d_\beta:\alpha\neq\beta<\lambda\}.$$

First note that this theory is consistent. Indeed, any finite part of T' uses only finitely many axioms of the form $d_\alpha\neq d_\beta$. Thus if we interpret $c_a^M=a$ and d_α's appearing in the finite fragment as any distinct elements (here we use the infinitude of M), we get consistency of that part. Using the completeness theorem, T' has a model. Either by our proof of completeness, or by a), we get that T' has a model N of cardinality exactly λ. We define $F:M\to N$ by $F(a)=c_a^N$. Because of the axioms $\varphi(c_{a_1},\ldots,c_{a_n})$ in T', we get that F is an elementary embedding. □

Note that in the upper Löwenheim–Skolem theorem we can construct $M \prec N'$, with N' of cardinality λ, by extending $F : M \to N$ to a bijection $F' : N' \to N$ (where N' is any set of cardinality λ containing M) and transporting back the L-structure from N to N' via the bijection F'.

For an L-structure M, we denote by $Th(M)$ the set of L-sentences true in M - we call it the *complete theory of* M. If $A \subseteq M$ is a subset (that can be the whole M) and $L' = L \cup \{c_a : a \in A\}$ and

$$\mathrm{Diag}_{\mathrm{el}}(A) = \{\varphi(c_{a_1}, \dots, c_{a_n}) : \text{ if } M \models \varphi(a_1, \dots, a_n)\}$$

we call $\mathrm{Diag}_{\mathrm{el}}(A)$ the *elementary diagram of* A. Similarly, we call

$$\mathrm{Diag}(A) = \{\varphi(c_{a_1}, \dots, c_{a_n}) : \text{ if } M \models \varphi(a_1, \dots, a_n) \text{ and } \varphi \text{ is } (\neg)\text{atomic}\}$$

the *atomic diagram of* A (we include sentences that come from atomic formulas and their negations). We call a formula φ *quantifier-free* if in its inductive definition no quantifiers are used. Also, we call a formula *universal* (resp. *existential*) if it is of the form $\forall \overline{x}\ \varphi$ (resp. $\exists \overline{x}\ \varphi$) for some quantifier-free formula φ. We call a formula an $\forall\exists$-*formula* if it is of the form $\forall \overline{x}\ \exists \overline{y}\ \varphi$, where φ is quantifier-free.

Exercises, sheet 2. Fix a first-order language L. M and N will denote L-structures and T will denote an L-theory. If $(I, <)$ is a linear order, by a (an elementary) chain of L-structures we mean a collection $(M_i)_{i \in I}$ such that $M_i \subseteq M_j$ if $i < j$ (resp. $M_i \prec M_j$).

1. Show that any for any finite structure in a finite language there is a sentence that determines it up to an isomorphism.
2. Show for an elementary chain $(M_i)_{i \in I}$, each M_i is elementarily embedded in the union of all M_j's (define a natural L-structure on this union). Generalise to the case where $(I, <)$ is a directed system.
3. For a theory T let $T_\forall$ be the set of universal sentences that are consequences of T. Show that $M \models T_\forall$ if and only if there is an $N \models T$ with $M \subseteq N$.
4. Deduce that a formula $\varphi(\overline{x})$ is T-equivalent to a universal formula if and only if for every $N \subseteq M$, $M \models T$ and $\overline{a} \in N^{|\overline{x}|}$, we have
$$M \models \varphi(\overline{a}) \implies N \models \varphi(\overline{a}).$$
5. Show that $\mathrm{Diag}_{\mathrm{el}}(M)$ (resp. $\mathrm{Diag}(M)$) is consistent with T, then there is an elementary embedding (resp. an embedding of L-structures) of M in a model of T.
6. We call T *model complete* if for every model $M \models T$, its atomic diagram is a complete theory. Show that the following are equivalent:
   - T is model complete,
   - every embedding of models of T is elementary,
   - every formula is T-equivalent to a universal one,
   - every formula is T-equivalent to an existential one.
7. Show that a theory T can be axiomatized by $\forall\exists$-formulas (i.e., T is *inductive*) if and only if for any chain of L-structures $(M_i)_{i \in I}$ that are models of T, their union is also a model of T.
8. A model $M \models T$ is called *existentially closed* if for any extension $M \subseteq N$ with $N \models T$, M is *existentially closed in* N (written $M \prec_\exists N$), i.e., whenever $\varphi(\overline{y})$ is an existential formula and $\overline{b} \in M^{|\overline{y}|}$, then
$$M \models \varphi(\overline{b}) \iff N \models \varphi(\overline{b}).$$

Show that if T is inductive, then any model of T can be embedded in an existentially closed model of T. Show that T is model complete if and only if all of its models are existentially closed.

9. An L-theory $T^\star$ is a *model companion* of T if:
   - every model of T admits an embedding into a model of $T^\star$,
   - every model of $T^\star$ admits an embedding into a model of T,
   - $T^\star$ is model complete.

   Show that if a model companion of T exists, it is unique up to equivalence of theories.
10. Show that if T is inductive, its model companion exists if and only if the class of its existentially closed models is axiomatized by a theory (i.e., this class is *elementary*).

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