

MODEL THEORY

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1. INTRODUCTION

Model theory is a part of logic where one studies classes of models of various sets of axioms. The fundamental questions of the area are: how to classify/construct models, what configurations can appear in a model, and how are different theories (sets of axioms) connected to each other. The standard methods of model theory are set up in the framework of first-order logic, i.e., one can only take quantifiers over elements, not over subsets. In applications it is often a non-trivial task to translate a problem into a first-order equivalent, but when possible it can lead to strong results.

Every theory is expressed in a language, e.g., groups have group multiplication, rings have $+$, $\cdot$, and vector spaces over a field have $+$ and multiplications by scalars. Equality is assumed to be in the language by default.

Example 1.1. Consider the theory

$$T = \left\{ (\exists x_1, \dots, x_n) \left(\bigwedge_{i \neq j} x_i \neq x_j \right) : n \in \mathbb{N} \right\}.$$

Models of T are just infinite sets $M \models T$. Up to isomorphism, these are classified by their cardinality. For a formula

$$\varphi(\bar{x}) = \exists y_1 \forall y_2 \dots \exists y_{2n-1} \forall y_{2n} \bigwedge_i \bigvee_j (\neg) \alpha_{ij} = \beta_{ij},$$

where $\alpha_{ij}, \beta_{ij} \in \{x_a, y_b : a, b\}$, one can show ⑥ that $\varphi(M) \subseteq M^{|\bar{x}|}$ is a Boolean combination of weak diagonals.

Of course, some properties of infinite sets depend on the model of ZFC we are working with, but we will treat the cardinal hierarchy as absolute, and rather reduce questions about models to, for example, cardinality questions. Surprisingly, many properties of the class of models of a theory are independent of the model of ZFC we choose. For a discussion on why we are working with a metatheory, see [3, 6].

For example, regardless of the universe of set theory, for the above theory T its spectrum function

$$I(T, \kappa) := |\{M \models T : |M| = \kappa\}| / \text{iso.}$$

is constantly equal to 1. A theory T such that $I(T, \kappa) = 1$ is called κ -categorical, and characterisation of such theories (for $\kappa = \aleph_0$ or $\kappa > \aleph_0$) is one of the most fundamental results in model theory (Ryll-Nardzewski's and Morley's theorem, respectively).

The scope of model theory has increased substantially since its origin. The bounds of the area are ultimately defined by what its users can prove. Nonetheless, there are several slogans describing what model theory is, that perhaps reflect the history of the development of the subject:

[Chang and Keisler, 1973] Model theory = Logic + Universal algebra,

[Hodges, 1997] Model theory = Algebraic geometry - Fields,

[Hrushovski] Model theory is the geography of tame mathematics.

These will hopefully become more meaningful during the course-time. The term “geography” in Hrushovski's description may be loosely treated as a reference to Shelah's idea of drawing division lines on the map of all theories: defining properties of theories that reflect the structure of their models. This point of view is exemplified quite literally in the model theoretic “map of the universe” [2].

Each ‘geographical’ region has a different culture and can lead to different interesting applications. During the course we can venture into various model-theoretic continents including frameworks for:

- geometry of differential equations and differential Galois theory (stability);
- difference equations, pseudo-finite fields (simplicity);
- tame topology and real-algebraic geometry (o-minimality);
- valued fields, tame non-Archimedean geometry (NIP, generic stability).

Depending on the interests of the audience we will choose one or more directions and from time to time we will supplement examples from the corresponding regions.

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2. COMPLETENESS THEOREM

The fundamental tool of model theory is Gödel's completeness theorem. It was proven by Kurt Gödel in his PhD thesis in 1929 and reformulated in Leon Henkin's PhD thesis in 1947. The theorem says that every first-order statement that is true in every model of some theory T admits a proof from the axioms of T . To state the completeness theorem formally we need to set the stage.

Definition 2.1. A language L consists of collections of *function symbols* $\{f_i\}_{i \in I}$, *relation symbols* $\{P_j\}_{j \in J}$, and *constant symbols* $\{c_k\}_{k \in K}$. Furthermore, each function and relation symbol is equipped with its *arity*, which is a natural number n_i for $i \in I$, m_j for $j \in J$.

Fix a language L for the rest of this section.

Definition 2.2. An L -structure is a non-empty set M together with *interpretations* of:

- functions $f_i^M : M^{n_i} \rightarrow M$ for $i \in I$,
- relations $P_j^M \subseteq M^{m_j}$ for $j \in J$,
- constants $c_k^M \in M$ for $k \in K$.

Definition 2.3. Fix an infinite set of variables $\{v\}_{v \in V}$. The L -terms (in variables $\{v\}_{v \in V}$) are elements of the smallest set that contains the variables, constant symbols, and whenever $i \in I$, $t_1, \dots, t_{n_i}$ are terms, then so is $f_i(t_1, \dots, t_{n_i})$.

We define the set of L -formulas (in variables $\{v\}_{v \in V}$) inductively as follows:

- An *atomic formula* is either of the form $t_1 = t_2$, where t_1, t_2 are terms in variables $\{v\}_{v \in V}$, or of the form $P_j(t_1, \dots, t_{m_j})$ for some $j \in J$ and terms $t_1, \dots, t_{m_j}$.
- If φ is a formula, then so is $\neg\varphi$.
- If φ, ψ are formulas, then so are $\varphi \wedge \psi, \varphi \vee \psi$.
- If φ is a formula and $v \in V$, then so are $(\exists v)(\varphi)$ and $(\forall v)(\varphi)$.

It would be enough to use e.g. $\wedge, \neg, \exists$ as logical symbols, and define other ones as combinations of those, for example $(\forall x)(\varphi) := \neg(\exists x)(\neg\varphi)$. Similarly, we use the conventions $(\varphi \vee \psi) := \neg(\neg\varphi \wedge \neg\psi)$, $(\varphi \rightarrow \psi) := (\neg\varphi \vee \psi)$ and $(\varphi \leftrightarrow \psi) := (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$. Also, we write $\bigwedge_{i=1}^l \varphi_i, \bigvee_{i=1}^l \varphi_i$ for the inductively defined iterations of $\wedge, \vee$ respectively. We often skip parentheses, L and variables in the names, if

those are clear from the context. We fix an infinite set of variables $\{v\}_{v \in V}$ for the rest of this section.

For every term t and formula φ we inductively define the set of variables $FV(t)$, $FV(\varphi)$ occurring freely in t , φ respectively, so that:

- (i) $FV(v) = \{v\}$, $FV(c_k) = \emptyset$, $FV(f_i(t_1, \dots, t_{n_i})) = \bigcup_{a=1}^{n_i} FV(t_a)$.
- (ii) $FV(\neg\varphi) = FV(\varphi)$, $FV(\varphi \wedge \psi) = FV(\varphi) \cup FV(\psi)$.
- (iii) $FV(\exists v \varphi) = FV(\varphi) \setminus \{v\}$.

We often denote variables by $x_1, \dots, x_n$. We write $t(x_1, \dots, x_n)$ or $\varphi(x_1, \dots, x_n)$ to mean that the free variables of t or φ respectively, are among $x_1, \dots, x_n$. We use overlines as tuples of variables $\bar{x} = (x_1, \dots, x_n)$.

Definition 2.4. A formula without free variables is called a *sentence*. A set of sentences is called an *L-theory*.

If M is an L -structure and $t(x_1, \dots, x_n)$ is a term, we inductively define the interpretation of t as a function $t^M : M^n \rightarrow M$, or as an element of M if $FV(t) = \emptyset$. If t is a single variable or a constant c_k , t^M is the appropriate projection function $M^n \rightarrow M$ or $c_k^M \in M$ respectively. If $t = f_i(t_1, \dots, t_{n_i})$, then t^M is defined by applying f_i^M to the values of $t_1^M, \dots, t_{n_i}^M$ on the appropriate tuple of elements of M . The precise definition of this step is left as an exercise ©.

Definition 2.5 (Tarski's definition of truth). Let M be an L -structure. We define a relation $M \models \varphi(a_1, \dots, a_n)$ for formulas $\varphi(x_1, \dots, x_n)$ and (possibly empty) tuples $\bar{a} = (a_1, \dots, a_n) \in M^n$ inductively as follows:

- (1) If $\varphi(\bar{x})$ is $t_1(\bar{x}) = t_2(\bar{x})$, then $M \models \varphi(\bar{a})$ if $t_1^M(\bar{a}) = t_2^M(\bar{a})$.
- (2) If $\varphi(\bar{x})$ is $P_j(t_1(\bar{x}), \dots, t_{m_j}(\bar{x}))$ then $M \models \varphi(\bar{a})$ if $(t_1^M(\bar{a}), \dots, t_{m_j}^M(\bar{a})) \in P_j^M$.
- (3) $M \models \neg\varphi(\bar{a})$ if it is not the case that $M \models \varphi(\bar{a})$ (written $M \not\models \varphi(\bar{a})$).
- (4) $M \models (\varphi \wedge \psi)(\bar{a})$ if $M \models \varphi(\bar{a})$ and $M \models \psi(\bar{a})$.
- (5) If $\varphi(\bar{x}) = (\exists y)(\psi(\bar{x}, y))$, then $M \models \varphi(\bar{a})$ if there is $b \in M$ such that $M \models \psi(\bar{a}, b)$.

If $M \models \varphi(\bar{a})$, we say that M *models* $\varphi(\bar{a})$. If ψ has free variables $x_1, \dots, x_n$, by convention we define $M \models \psi$ to mean $M \models \bar{\psi}$, where $\bar{\psi} = \forall x_1 \dots \forall x_n \psi$ is the *universal closure* of ψ . For an L -theory T we say that an L -structure M is a *model* of T , written $M \models T$, if for every $\varphi \in T$, $M \models \varphi$.

In order to state Gödel's completeness theorem it remains to define the notion of a proof. First, recall that a *propositional formula* is an expression built using propositional symbols $P_1, P_2, \dots$ and Boolean connectives $\neg, \wedge$ (we can also add $\vee, \rightarrow, \leftrightarrow$, or define them similarly as before). We say that a propositional formula is a *tautology*, if under any substitution of *Truth* and *False* for P_i 's, the resulting term in the Boolean algebra $\{\text{Truth}, \text{False}\}$ is evaluated as *Truth*. If $\tau(P_1, \dots, P_n)$ is a propositional formula, and $\varphi_1, \dots, \varphi_n$ are formulas, we call $\tau(\varphi_1, \dots, \varphi_n)$ an *instantiation* of τ .

Lemma 2.6. If $\tau(P_1, \dots, P_n)$ is a tautology and $\varphi_1, \dots, \varphi_n$ are formulas, then for all L -structures M we have

$$M \models \tau(\varphi_1, \dots, \varphi_n).$$

Proof. Exercise ©. □

Definition 2.7. The *axioms of the classical first-order logic* (in the language L and variables $\{v\}_{v \in V}$) consist of:

- A0. Instantiations $\tau(\varphi_1, \dots, \varphi_n)$, for tautologies τ and formulas $\varphi_1, \dots, \varphi_n$.
- A1. Formulas $\forall x(\varphi \rightarrow \psi) \rightarrow (\varphi \rightarrow \forall x \psi)$, for all pairs of formulas φ, ψ , such that x is not free in φ .
- A2. Formulas $\forall x \varphi \rightarrow \varphi(x/t)$, for all formulas φ and terms t such that none of the free occurrences of x in φ is in the range of a quantifier bounding a free variable from t . Here $\varphi(x/t)$ is the result of substitution of every free occurrence of x in φ by t .
- A3. Equality axioms:
 - $x = x$ for every variable x ;
 - $x = y \rightarrow t(\dots, x, \dots) = t(\dots, y, \dots)$, for every term t and every pair of variables x, y , where y appears in the same place in t as x was;
 - $x = y \rightarrow (\varphi(\dots, x, \dots) \leftrightarrow \varphi(\dots, y, \dots))$, for every formula φ and every pair of variables x, y , where y appears in the same place in φ as x was, and x occurs freely in φ , not in the range of a quantifier bounding y .

To be extremely formal in the above definition we would have to define $\varphi(x/t)$ and conditions like “none of the occurrences of x in φ is in the range of a quantifier bounding a free variable from t ” inductively. We leave it as an exercise ⑥ to the reader. If $\varphi(\bar{x})$ is a formula and $\bar{c}$ is tuple of constants of the same length, then we write $\varphi(\bar{c})$ for the substitution of respective x_i ’s by c_i ’s. Note that the restriction in A2 makes sense, as if $\varphi = (\exists y)(x \neq y)$ and $t = y$, then it should not be an axiom that

$$\forall x \exists y x \neq y \rightarrow (\exists y)(y \neq y).$$

Definition 2.8. The *deduction rules of the classical first-order logic* consist of the modus ponens (MP) rule

$$\frac{\varphi, \varphi \rightarrow \psi}{\psi}$$

and the $\forall$ -rule

$$\frac{\varphi}{\forall x \varphi},$$

for every pair of formulas φ, ψ .

Definition 2.9. We say that a set of formulas X proves a formula ψ , written $X \vdash \psi$, if there is a finite sequence of formulas $\alpha_1, \dots, \alpha_n = \psi$, such that for every $i = 1, \dots, n$ one of the following two conditions holds:

- (i) α_i is an axiom of the classical first-order logic or an element of X .
- (ii) α_i can be deduced from $\alpha_1, \dots, \alpha_{i-1}$ using one of the deduction rules. More precisely, there are $j, k < i$ such that $\alpha_k = (\alpha_j \rightarrow \alpha_i)$, or $\alpha_i = \forall x \alpha_k$.

If $X = \emptyset$ in the above definition, we just write $\vdash \psi$ instead of $\emptyset \vdash \psi$.

Example 2.10. Let us show that $\vdash \forall x \psi \rightarrow \exists x \psi$. Let β be the formula we want to prove and let y be a variable not appearing in β . The proof can be constructed as follows:

- α_1 : $\forall x \psi \rightarrow \psi(x/y)$, by A2.
- α_2 : $\forall x \neg \psi \rightarrow \neg \psi(x/y)$, by A2.
- α_3 : $\alpha_2 \rightarrow (\psi(x/y) \rightarrow \neg \forall x \neg \psi)$, by A0.
- α_4 : $\psi(x/y) \rightarrow \neg \forall x \neg \psi$, by MP.
- α_5 : $\alpha_1 \rightarrow (\alpha_4 \rightarrow \beta)$, by A0.

$\alpha_6: \alpha_4 \rightarrow \beta$, by MP.

$\alpha_7: \beta$ by MP.

Exercise 2.11 (Deduction theorem). ⑥ Let φ be a sentence, ψ be a formula, and X be a set of formulas. Then:

$$X \vdash \varphi \rightarrow \psi \iff X \cup \{\varphi\} \vdash \psi.$$

Exercise 2.12. ⑥ For any formula φ we have $\varphi \vdash \bar{\varphi}$ and $\bar{\varphi} \vdash \varphi$.

For a formula φ and a theory T , we write $T \models \varphi$, if for every L -structure $M \models T$ we have $M \models \varphi$. If $T = \emptyset$ we abbreviate it as $\models \varphi$.

Theorem 2.13 (Soundness). Let φ be a formula. If $\vdash \varphi$, then $\models \varphi$.

Proof. By induction, left as an exercise. ⑥. □

Note that by the finitary nature of proofs and the deduction lemma, it follows that for a theory T , $T \vdash \psi$ implies that $T \models \psi$.

Theorem 2.14 (Gödel's completeness). Let φ be a formula. If $\models \varphi$, then $\vdash \varphi$.

A set of formulas S is *consistent*, if $S \not\vdash (\exists x)(x \neq x)$.

Theorem 2.15 (Completeness after Henkin). Let S be a consistent set of sentences. Then S has a model.

Proof. Let κ be the maximum of cardinalities of the number of variables and size of the language. Consider a new language L' which has the same symbols as L together with κ -many new constant symbols $(d_\lambda)_{\lambda < \kappa}$. Let $(\varphi_i(x))_{i < \kappa}$ be a sequence of L' -formulas with one free variable x . Let $g : \kappa \rightarrow \kappa$ be an increasing function such that for every $\lambda < \kappa$, $d_{g(\lambda)}$ does not appear in $\varphi_i(x)$ for $i \leq \lambda$. Now consider the set of L' -sentences

$$S' = S \cup \{\exists x \varphi_i(x) \rightarrow \varphi_i(d_{g(i)}) : i < \kappa\}.$$

We claim that S' is consistent. Indeed, if it were not, there would be $i_1 < \dots < i_n$ such that

$$S'' = S \cup \{\exists x \varphi_{i_j}(x) \rightarrow \varphi_{i_j}(d_{g(i_j)}) : j = 1, \dots, n-1\}$$

is consistent, and it becomes inconsistent if $\exists x \varphi_{i_n}(x) \rightarrow \varphi_{i_n}(d_{g(i_n)})$ is added to it. Thus, $S'' \vdash \exists x \varphi_{i_n}(x) \wedge \neg \varphi_{i_n}(d_{g(i_n)})$. In particular $S'' \vdash \neg \varphi(d_{g(i_n)})$. By the definition of $g(i_n)$, $d_{g(i_n)}$ does not appear in formulas from S'' . Hence, we can rewrite the proof witnessing that $S'' \vdash \neg \varphi_{i_n}(d_{g(i_n)})$, so that we get $S'' \vdash \neg \varphi_{i_n}(x)$. This contradicts the fact that $S'' \vdash \exists x \varphi_{i_n}(x)$.

Let T be a maximal consistent L' -theory containing S' . Such an extension exists by the Kuratowski-Zorn lemma. It is easy to see that for every L' -sentence φ , we have $\varphi \in T$ or $\neg \varphi \in T$ (theories with this property are called *complete*). We consider a relation on the set of constants $\{d_\lambda\}_{\lambda < \kappa}$ defined by

$$d_\lambda \sim d_\mu \iff T \vdash d_\lambda = d_\mu.$$

It is an equivalence relation and we define an L -structure M by considering the set of equivalence classes of $\sim$, and by requiring that:

- (1) $f^M([d_{i_1}]_\sim, \dots, [d_{i_n}]_\sim) = [d_j]_\sim \iff T \vdash f(d_{i_1}, \dots, d_{i_n}) = d_j$ for every function symbol f of arity n in L ;
- (2) $([d_{i_1}]_\sim, \dots, [d_{i_m}]_\sim) \in R^M \iff T \vdash R(d_{i_1}, \dots, d_{i_m})$ for every relation symbol R of arity m in L ;

(3) $c^M = [d_j]_\sim \iff T \vdash c = d_j$, for every constant symbol c in L .

We leave the proof of the fact that these are well defined and do not depend on choices for the reader ⑥. The existence of the d_j in the cases (1), (3) follows from the fact that for every term t without free variables, in any language, $\vdash \exists x t = x$, which we leave as an exercise ⑥. It remains to show that $M \models \varphi$, for $\varphi \in S$. More generally, we will show that

$$M \models \alpha \iff T \vdash \alpha,$$

for all L' -sentences (not formulas), where we interpret d_i^M as $[d_i]_\sim$. We do it by induction on formula complexity of α :

- If α is an atomic L' -formula, then it must be of the form $R(t_1, \dots, t_m)$ or $t_1 = t_2$, for some terms $t_1, t_2, \dots, t_m$ and a relation symbol R of arity m . If $t_i^M = [d_{\lambda_i}]_\sim$ for all i , then $T \vdash t_i = d_{\lambda_i}$ for all i , so e.g.

$$M \models \alpha \iff M \models R([d_{\lambda_1}]_\sim, \dots, [d_{\lambda_m}]_\sim)$$

$$\iff T \vdash R(d_{\lambda_1}, \dots, d_{\lambda_m}) \iff T \vdash R(t_1, \dots, t_m).$$

The case of equality of two terms is the same.

- If α is a conjunction then the equivalence is obvious. If $\alpha = \neg\beta$, then

$$M \models \alpha \iff M \not\models \beta \iff T \not\vdash \beta \iff T \vdash \neg\beta,$$

where the last equivalence uses the fact that T is maximal consistent.

- If $\alpha = \exists x \varphi_i(x)$, then we choose a witness $[d_\lambda]_\sim$ and we get

$$M \models \varphi_i([d_\lambda]_\sim) \iff M \models \varphi_i(d_\lambda) \iff T \vdash \varphi_i(d_\lambda) \implies T \vdash \exists x \varphi_i(x).$$

The middle equivalence follows from the inductive hypothesis. On the other hand

$$T \vdash \exists x \varphi_i(x) \implies T \vdash \varphi_i(d_{g(i)}) \iff M \models \varphi_i(d_{g(i)})$$

$$\iff M \models \varphi_i([d_{g(i)}]_\sim) \implies M \models \exists x \varphi_i(x).$$

Here the first equivalence follows from the inductive hypothesis.

This completes the proof, as $S \subseteq T$. $\square$

Henkin's proof of completeness is important as it provides a step-by-step 'construction'. It can be generalised to construct models with some specific non-first-order properties.

Corollary 2.16. ⑥ Let T be a theory and ψ be a sentence. Then

$$T \models \psi \iff T \vdash \psi.$$

Corollary 2.17 (Compactness theorem). ⑥ Let T be a theory. Then T has a model if and only if for every finite $T_0 \subseteq T$, T_0 has a model.

From now on, we identify theories that have the same logical consequences, i.e., when we write $T = T'$, we mean $T \vdash T'$ and $T' \vdash T$, or equivalently that for any L -structure M , $M \models T \iff M \models T'$.

Exercises, sheet 1.

1. Fill the gaps left in the proofs above. Exercises throughout the text are marked with the symbol ©. Read the proof of completeness theorem carefully.
2. Prove that if X is a set of formulas and $X \vdash \varphi(c)$, where c is a constant not appearing in any formula from X and $\varphi(x)$ is a formula with a free variable x , then $X \vdash \forall x \varphi(x)$.
3. Show that in any language for any formula φ there is a prenex normal form formula ξ such that $\vdash \varphi \leftrightarrow \xi$.
4. Let $L = (+, \cdot, <, (c_r)_{r \in \mathbb{R}}, \dots)$, where in (...) we include a unary (i.e., of arity one) function symbol for every function $f : \mathbb{R} \rightarrow \mathbb{R}$. Consider $\mathbb{R}$ as an L -structure. Show that there is a model M of $T = Th(\mathbb{R}) = \{\sigma : \mathbb{R} \models \sigma\}$ such that in M there exists $\varepsilon > 0$ with $\varepsilon < c_r^M$ for all $r \in \mathbb{R}_{>0}$. Hint: use compactness theorem for some $L' = L \cup \{\epsilon\}$ -theory and put $\varepsilon = \epsilon^M$.
5. Note that the model M from the previous exercise contains an isomorphic copy of $\mathbb{R}$ (even as an L -structure). We will identify this copy with $\mathbb{R}$ itself.
6. Let $\langle \mathbb{R} \rangle \subseteq M$ be the convex hull of $\mathbb{R} \subseteq M$. Note that $x \in \langle \mathbb{R} \rangle$ if and only if $-n < x < n$ for some $n \in \mathbb{N}$. Show that for any such $x \in \langle \mathbb{R} \rangle \subseteq M$ there is a unique $r \in \mathbb{R}$ such that for all $m \in \mathbb{N}$ we have $-1/m < x - r < 1/m$. We denote it by $r = st(x)$.
7. Prove that $st : \langle \mathbb{R} \rangle \rightarrow \mathbb{R}$ has the following properties: $st(x + y) = st(x) + st(y)$, $st(xy) = st(x)st(y)$. Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous function and $r \in \mathbb{R}$. Show that $st(g^M(r + \varepsilon)) = g(r)$.
8. Let f be a function symbol corresponding to a differentiable function and let f' be the function symbol corresponding to its derivative. Show that for $r \in \mathbb{R}$ we have

$$st \left(\frac{f^M(r + \varepsilon) - f^M(r)}{\varepsilon} \right) = f'(r).$$

In particular, show that the term inside st is in $\langle \mathbb{R} \rangle$.

9. Prove the Leibniz rule $(fg)' = f'g + fg'$ using the above interpretation of the derivative.
10. Read the exercises below. Familiarise yourself with their statements. Their solutions are optional.

Fix a countable first-order language L . The set of L -formulas (in countably many variables) is sometimes denoted by $L_{\omega, \omega}$. This is to indicate that when we form L -formulas, we can only consider finite conjunctions $\bigwedge_{i=1}^n \varphi_i$ (with $n < \omega$) and finite quantifications $\exists x_1 \dots \exists x_m \varphi$ (with $m < \omega$). Given infinite cardinals κ, λ with $\lambda \leq \kappa$ and a set V of variables with $|V| = \kappa$ one can consider $L_{\kappa, \lambda}$ -preformulas which are defined similarly as $L_{\omega, \omega}$ -formulas (so we start with atomic formulas and have negation) but with the following connectives and quantification steps:

- If $\{\varphi_\alpha\}_{\alpha \in A}$ are $L_{\kappa, \lambda}$ -preformulas with $|A| < \kappa$ then so are

$$\bigwedge_{\alpha \in A} \varphi_\alpha, \bigvee_{\alpha \in A} \varphi_\alpha.$$

- If φ is a $L_{\kappa, \lambda}$ -preformula and $X \subseteq V$ satisfies $|X| < \lambda$, then so is $\exists X \varphi$.

We distinguish among those $L_{\kappa, \lambda}$ -formulas as those that have only fewer than λ free variables. In the following exercise we will be interested in $L_{\omega_1, \omega}$. In the proof

system for $L_{\omega_1, \omega}$ we add axioms

$$\bigwedge_{\alpha \in A} \varphi_\alpha \rightarrow \varphi_{\alpha_0},$$

for any countable collection of formulas $\{\varphi_\alpha\}_{\alpha \in A}$ whose conjunction is a formula, and $\alpha_0 \in A$. Furthermore we add the deduction rule scheme

$$\frac{\varphi_0, \varphi_1, \dots}{\bigwedge_{i \in \mathbb{N}} \varphi_i},$$

for every sequence of $L_{\omega_1, \omega}$ -formulas $\{\varphi_i\}_{i \in \mathbb{N}}$ whose conjunction is a formula. Here proofs are infinitely branching trees with finite depth branches.

Exercise 2.18. (e) Imitate Henkin's proof to show that if an $L_{\omega_1, \omega}$ -sentence σ is consistent, then it has a model. Conclude with the completeness theorem for infinitary formulas $\models \varphi \iff \vdash \varphi$.

Some systems of axioms use more than one type of primitive object. For example, Euclid's axioms are about points P , lines l , and the incidence relation between them $P \in l$. One can formalise this in the framework of multisorted (first-order) logic, where a language L consists of:

- Sort symbols $\{S_\alpha\}_{\alpha \in A}$.
- Function symbols $\{f_i\}_{i \in I}$, each equipped with its arity, which is a tuple $(\alpha_1, \dots, \alpha_n; \beta)$ for some $n \in \mathbb{N}$ and $\alpha_1, \dots, \alpha_n, \beta \in A$.
- Predicate symbols $\{P_j\}_{j \in J}$, each equipped with its arity which is a tuple $(\alpha_1, \dots, \alpha_m)$ for some $m \in \mathbb{N}$ and $\alpha_1, \dots, \alpha_m \in A$.
- Constant symbols $\{c_k\}_{k \in K}$, each equipped with a sort of index $\alpha = \alpha(k) \in A$.

In that case, an L -structure M is a collection of non-empty sets S_α^M together with interpretations of other symbols, where if f is a function symbol with arity $(\alpha_1, \dots, \alpha_n; \beta)$, then its interpretation is a function

$$f^M : S_{\alpha_1}^M \times \dots \times S_{\alpha_n}^M \rightarrow S_\beta^M.$$

Similarly, if P is a predicate symbol with arity $(\alpha_1, \dots, \alpha_m)$, then its interpretation is a subset

$$P^M \subseteq S_{\alpha_1}^M \times \dots \times S_{\alpha_m}^M.$$

If c is a constant of sort S , then M is equipped with its interpretation $c^M \in S^M$. When defining L -formulas, we need to consider variables decorated by sort symbols so each sort has a different set of variables associated to it.

Exercise 2.19. (e) Convince yourself that Henkin's proof of completeness works for multisorted structures.

Exercise 2.20. (e) Look at Hilbert's axioms of Euclidean geometry and define an appropriate multisorted first-order language.

Many axiom systems involve a complete metric and so the real numbers in their definition. For example the axioms of Banach spaces or Hilbert spaces are such. Continuous (first-order) logic is a variant of first-order logic, where one can consider some real-valued predicates and the equality is replaced by the metric. In the following by a *continuity modulus* we mean a continuous increasing function

$$\delta : [0, 1] \rightarrow [0, 1]$$

satisfying $\delta(0) = 0$ and $\delta(\varepsilon) > 0$ for $\varepsilon > 0$.

Definition 2.21. A *continuous language* L consists of:

- sort symbols S each equipped with a metric symbol d_S and with a bound $B_S \in \mathbb{R}_{\geq 0}$;
- constant symbols c with their arity as in the multisorted logic;
- function symbols f with the same arities as in the multisorted logic, each equipped with its continuity modulus δ_f ;
- predicate symbols P with their arities same as in the multisorted logic, each equipped with its continuity modulus δ_P and with a bound $B_P \in \mathbb{R}_{\geq 0}$.

An L -structure M consists of interpretations of sorts, constant and function symbols, same as in multisorted logic, however, for each sort S its interpretation S^M is equipped with the metric symbol interpretation d_S^M which makes (S^M, d_S^M) a non-empty complete metric space of diameter at most B_S . Moreover, the function symbols interpretations are supposed to respect the continuity modulus in the following sense: for all real ε with $0 < \varepsilon \leq 1$, for every $a_1, \dots, a_n, b_1, \dots, b_n$ in appropriate sorts of M , we have

$$\max_{i=1, \dots, n} d(a_i, b_i) < \delta_f(\varepsilon) \implies d(f(a_1, \dots, a_n), f(b_1, \dots, b_n)) \leq \varepsilon,$$

where we skipped interpretation in M (and the indexes for sorts) in the notation. Furthermore, the interpretations of predicate symbols are functions

$$P^M : S_{\alpha_1}^M \times \dots \times S_{\alpha_m}^M \rightarrow [0, B_P]$$

that are required to respect the continuity modulus in the following sense: for all real $1 \geq \varepsilon > 0$ for every $a_1, \dots, a_m, b_1, \dots, b_m$ in appropriate sorts of M we have

$$\max_{i=1, \dots, m} d(a_i, b_i) < \delta_P(\varepsilon) \implies |P(a_1, \dots, a_m) - P(b_1, \dots, b_m)| \leq \varepsilon.$$

When we build formulas in continuous logic, terms are defined similarly as in the discrete case. More complicated expressions are constructed by:

- atomic formulas - if t_1, t_2 are terms of the same sort S , then $d_S(t_1, t_2)$ is an atomic formula. If $t_1, \dots, t_n$ are terms of appropriate sorts, and P is a predicate symbol whose arity has matching sorts, then $P(t_1, \dots, t_n)$ is an atomic formula.
- connectives - if $\varphi_1, \dots, \varphi_n$ are formulas and $u : \mathbb{R}^n \rightarrow \mathbb{R}$ is a continuous function, then $u(\varphi_1, \dots, \varphi_n)$ is a formula.
- quantifiers - if φ is a formula then so are $\sup_x \varphi$ and $\inf_x \varphi$.

In some variants of continuous logic one restricts only to $[0, 1]$ -valued formulas (all the bounds are equal to 1), and connectives are taken to be only $x \mapsto 1 - x, x \mapsto x/2, (x, y) \mapsto \max(x - y, 0)$. If M is an L -structure, $\varphi(\bar{x})$ is a formula with free-variables $\bar{x}$ and $\bar{a} \in M^{\bar{x}}$, then one inductively defines $\varphi(\bar{a})^M \in \mathbb{R}$. A sentence is said to hold in M if it is evaluated to zero. It is possible to formalise the notion of a proof in continuous logic so that a completeness theorem holds, see [1]. Later, we may prove the following theorem, using ultraproducts.

Theorem 2.22 (Compactness for continuous logic). Let T be a continuous logic theory. Assume that for every $\varphi_1, \dots, \varphi_n \in T$ and every $\varepsilon > 0$ there is an L -structure M such that for all $i = 1, \dots, n$ we have $|\varphi_i^M| \leq \varepsilon$. Then T has a model.

Exercise 2.23. © Axiomatize the following structures in continuous logic: Banach spaces, Hilbert spaces, C^* -algebras. Hint: introduce a sort for each ball of radius $n \in \mathbb{N}$.

3. LÖWENHEIM–SKOLEM THEOREM

Fix a first-order language L , so if we refer to formulas or sentences we mean the ones over L . We use letters M, N to denote L -structures.

Definition 3.1. (1) A *homomorphism between M and N* is a function $F : M \rightarrow N$ such that for every function symbol f of arity n , predicate symbol P of arity m , and constant symbol c we have

$$\begin{cases} F(f^M(a_1, \dots, a_n)) = f^N(F(a_1), \dots, F(a_n)), \\ P^M(b_1, \dots, b_m) \implies P^N(F(b_1), \dots, F(b_m)), \\ F(c^M) = c^N, \end{cases}$$

for any a_i 's and b_j 's in M . In particular, L -structures form a category, and an *isomorphism* is a homomorphism that has an inverse homomorphism. We write $M \cong N$ to denote that M and N are isomorphic L -structures.

- (2) We say that M is a *substructure of N* , written $M \subseteq N$, if corresponding sets are contained in each other, and interpretations of all symbols in N restrict to interpretations of these symbols in M .
- (3) We say that M is *elementarily equivalent to N* , written $M \equiv N$, if for every sentence φ we have $M \models \varphi \iff N \models \varphi$.
- (4) We say that M is an *elementary substructure of N* , written $M \prec N$, if M is a substructure of N and for any finite tuple $\bar{a} \subseteq M$ and any formula $\varphi(\bar{x})$ with $|\bar{x}| = |\bar{a}|$ we have

$$M \models \varphi(\bar{a}) \iff N \models \varphi(\bar{a}).$$

- (5) A function $\alpha : M \rightarrow N$ is *elementary* if for any finite tuple $a_1, \dots, a_n \in M$ and any formula $\varphi(\bar{x})$ with $|\bar{x}| = |\bar{a}|$ we have

$$M \models \varphi(a_1, \dots, a_n) \iff N \models \varphi(\alpha(a_1), \dots, \alpha(a_n)).$$

Theorem 3.2 (Tarski-Vaught test). Let A be a subset of N . Then A is a universe of an elementary substructure of N if and only the following condition $(\star)$ holds: For every formula $\varphi(x, \bar{y})$ and every $\bar{a} \in A^{|\bar{y}|}$ if $N \models \exists x \varphi(x, \bar{a})$, then there is $b \in A$ such that $N \models \varphi(b, \bar{a})$.

Proof. First, we assume that A is a universe of an elementary substructure. Then by the assumption

$$N \models \exists x \varphi(x, \bar{a}) \iff A \models \exists x \varphi(x, \bar{a}),$$

for $\varphi(x, \bar{y})$ and $\bar{a}$ as in $(\star)$, so there is $b \in A$ such that $A \models \varphi(b, \bar{a})$.

Now assume that $(\star)$ holds. Note that this implies that A is closed under N -interpretations of function symbols, as for a function symbol f of arity n , if $a_1, \dots, a_n \in A$, then by $(\star)$ we get

$$N \models \exists x f(a_1, \dots, a_n) = x \implies \text{there is } b \in A \text{ such that } f(a_1, \dots, a_n) = b.$$

Now we show that $A \prec N$ inductively, based on the complexity of the formula. Atomic formulas and connective steps are easy $\textcircled{e}$. Consider the formula $\varphi(\bar{y}) = \exists x \psi(x, \bar{y})$ and assume that ψ satisfies the inductive hypothesis. Let $\bar{a} \in A^{|\bar{y}|}$. We get

$$\begin{aligned} N \models \exists x \psi(x, \bar{a}) &\implies N \models \psi(b, \bar{a}) \text{ for some } b \in A \\ &\iff A \models \psi(b, \bar{a}) \text{ for some } b \in A \iff A \models \exists x \psi(x, \bar{a}), \end{aligned}$$

where the first implication follows by $(\star)$ and is actually an equivalence, and the middle equivalence follows from the inductive hypothesis. $\square$

By convention, the cardinality of the language is the cardinality of the set of formulas in countably many variables. Thus, it is maximum of the cardinality of the set of symbols and $\aleph_0$.

Corollary 3.3 (Löwenheim–Skolem theorem). Let M be an infinite L -structure and assume that $|L| = \kappa$. Then:

- a) There is an $N \prec M$ with $|N| \leq \kappa$.
- b) For any $\lambda \geq \max(\kappa, |M|)$ there is an L -structure N with $|N| = \lambda$ and with an elementary embedding $F : M \rightarrow N$.

Proof. a) Let $(\varphi_\alpha(x, \bar{y}))_{\alpha < \kappa}$ be an enumeration of all L -formulas where each formula appears cofinally many times. We define a sequence of subsets $(A_\alpha)_{\alpha < \kappa}$ inductively with $A_0 = \emptyset$ and A_β defined so that it contains $\bigcup_{\alpha < \beta} A_\alpha$ and for every $\alpha < \beta$ and every $\bar{a} \subseteq A_\alpha$ of the same length as $\bar{y}$ in $\varphi_\alpha(x, \bar{y})$, if $M \models \exists x \varphi_\alpha(x, \bar{a})$, then we choose a witness and add it to A_β . If $N = \bigcup_{\alpha < \kappa} A_\alpha$, then $|N| \leq \kappa$ and it satisfies the Tarski-Vaught test, so $N \prec M$.

- b) Let $L' = L \cup \{c_\alpha : \alpha \in M\} \cup \{d_\alpha : \alpha < \lambda\}$ and consider the L' -theory

$$T' = \{\varphi(c_{a_1}, \dots, c_{a_n}) : \text{if } M \models \varphi(a_1, \dots, a_n)\} \cup \{d_\alpha \neq d_\beta : \alpha \neq \beta < \lambda\}.$$

First note that this theory is consistent. Indeed, any finite part of T' uses only finitely many axioms of the form $d_\alpha \neq d_\beta$. Thus if we interpret $c_a^M = a$ and d_α 's appearing in the finite fragment as any distinct elements (here we use the infinitude of M), we get consistency of that part. Using the completeness theorem, T' has a model. Either by our proof of completeness, or by a), we get that T' has a model N of cardinality exactly λ . We define $F : M \rightarrow N$ by $F(a) = c_a^N$. Because of the axioms $\varphi(c_{a_1}, \dots, c_{a_n})$ in T' , we get that F is an elementary embedding. $\square$

Note that in the upper Löwenheim–Skolem theorem we can construct $M \prec N'$, with N' of cardinality λ , by extending $F : M \rightarrow N$ to a bijection $F' : N' \rightarrow N$ (where N' is some set of cardinality λ containing M) and transporting back the L -structure from N to N' via the bijection F' .

For an L -structure M , we denote by $Th(M)$ the set of L -sentences true in M - we call it the *complete theory of M* . If $A \subseteq M$ is a subset (that can be the whole M) and $L' = L \cup \{c_a : a \in A\}$ and

$$\text{Diag}_{\text{el}}(A) = \{\varphi(c_{a_1}, \dots, c_{a_n}) : \text{if } M \models \varphi(a_1, \dots, a_n)\}$$

we call $\text{Diag}_{\text{el}}(A)$ the *elementary diagram of A* . Similarly, we call

$$\text{Diag}(A) = \{\varphi(c_{a_1}, \dots, c_{a_n}) : \text{if } M \models \varphi(a_1, \dots, a_n) \text{ and } \varphi \text{ is } (\neg)\text{-atomic}\}$$

the *atomic diagram of A* (we include sentences that come from atomic formulas and their negations). We call a formula φ *quantifier-free* if in its inductive definition no quantifiers are used. Also, we call a formula *universal* (resp. *existential*) if it is of the form $\forall \bar{x} \varphi$ (resp. $\exists \bar{x} \varphi$) for some quantifier-free formula φ . We call a formula an $\forall\exists$ -*formula* if it is of the form $\forall \bar{x} \exists \bar{y} \varphi$, where φ is quantifier-free.

Exercises, sheet 2. Fix a first-order language L . M and N will denote L -structures and T will denote an L -theory. If $(I, <)$ is a linear order, by a (an elementary) chain of L -structures we mean a collection $(M_i)_{i \in I}$ such that $M_i \subseteq M_j$ if $i < j$ (resp. $M_i \prec M_j$).

1. Show that any for any finite structure in a finite language there is a sentence that determines it up to an isomorphism.
2. Show that isomorphic structures satisfy the same sentences. Hint: induction based on the complexity of the formula.
3. Show for an elementary chain $(M_i)_{i \in I}$, each M_i is elementarily embedded in the union of all M_j 's (define a natural L -structure on this union). Generalise to the case where $(I, <)$ is a directed system.
4. For a theory T let $T_\forall$ be the set of universal sentences that are consequences of T . Show that $M \models T_\forall$ if and only if there is an $N \models T$ with $M \subseteq N$.
5. Deduce that a formula $\varphi(\bar{x})$ is T -equivalent to a universal formula if and only if for every $N \subseteq M$, $M, N \models T$ and $\bar{a} \in N^{|\bar{x}|}$, we have

$$M \models \varphi(\bar{a}) \implies N \models \varphi(\bar{a}).$$

6. Show that $\text{Diag}_{\text{el}}(M)$ (resp. $\text{Diag}(M)$) is consistent with T , then there is an elementary embedding (resp. an embedding of L -structures) of M in a model of T .
7. We call T *model complete* if for every model $M \models T$, the $L(M)$ -theory $T \cup \text{Diag}(M)$ is complete (where $L(M) = L \cup \{c_m : m \in M\}$). Show that the following are equivalent:
 - T is model complete,
 - every embedding of models of T is elementary,
 - every formula is T -equivalent to a universal one,
 - every formula is T -equivalent to an existential one.
8. Show that a theory T can be axiomatized by $\forall\exists$ -formulas (i.e., T is *inductive*) if and only if for any chain of L -structures $(M_i)_{i \in I}$ that are models of T , their union is also a model of T .
9. A model $M \models T$ is called *existentially closed* if for any extension $M \subseteq N$ with $N \models T$, M is *existentially closed in* N (written $M \prec_\exists N$), i.e., whenever $\varphi(\bar{y})$ is an existential formula and $\bar{b} \in M^{|\bar{y}|}$, then

$$M \models \varphi(\bar{b}) \iff N \models \varphi(\bar{b}).$$

Show that if T is inductive, then any model of T can be embedded in an existentially closed model of T . Show that T is model complete if and only if all of its models are existentially closed.

10. An L -theory T^* is a *model companion* of T if:
 - every model of T admits an embedding into a model of T^* ,
 - every model of T^* admits an embedding into a model of T ,
 - T^* is model complete.

Show that if a model companion of T exists, it is unique up to equivalence of theories.

11. Show that if T is inductive, its model companion exists if and only if the class of its existentially closed models is axiomatized by a theory (i.e., this class is *elementary*).

4. CRITERIA FOR ELEMENTARY EQUIVALENCE

Fix a language L and let M, N be L -structures.

Question 4.1. How to check if $M \equiv N$?

There is a criterion for elementary equivalence in form of a game. Assume for simplicity that L is finite and has only relation symbols. It is a minor restriction as we can encode function symbols by their graphs and constants by new unary predicate symbols.

Definition 4.2. The *Ehrenfeucht–Fraïssé game of length $n \in \mathbb{N}$* , denoted by $\Gamma_n(M, N)$, is being played between two players I and II. The first player is called Spoiler and the second is called Duplicator.

- The game consists of rounds indexed by $i = 1, \dots, n$. At the i 'th round, elements $a_1, \dots, a_{i-1} \in M$ and $b_1, \dots, b_{i-1} \in N$ are already chosen.
- The player I does one of the following two things:
 - (1) either chooses $a_i \in M$, which is answered by a choice of $b_i \in N$ by the player II;
 - (2) or chooses $b_i \in N$, for which the player II answers with a choice of $a_i \in M$.
- At the end of the game the player II wins, if the resulting sequences $(a_1, \dots, a_n) \in M^n$ and $(b_1, \dots, b_n) \in N^n$ are such that the map $a_i \mapsto b_i$ is well defined and is a *partial isomorphism* of M and N , i.e., an isomorphism between L -substructures consisting of these n -tuples.

Example 4.3. Consider two linear orders: $(\mathbb{Z}, <)$ and $(\mathbb{Z} \sqcup \mathbb{Z}, <)$ (one copy after another). By playing “far enough” one can show (e) that II has a winning strategy in the length n Ehrenfeucht–Fraïssé game, for every $n \in \mathbb{N}$.

Theorem 4.4 (Ehrenfeucht–Fraïssé). M and N are elementarily equivalent if and only if II has a winning strategy in $\Gamma_n(M, N)$ for every $n \in \mathbb{N}$.

Proof. (e) Exercise. □

Let us come back to an arbitrary language L and an L -theory T .

Definition 4.5. Let κ be an infinite cardinal. A theory T is κ -categorical if it has a model of cardinality κ and for any two models $M, N \models T$ of cardinality κ , $M \cong N$.

Here is another criterion for elementary equivalence. Recall that due to our convention of identifying a theory with a set of its logical consequences, T is complete if for every sentence φ , $T \vdash \varphi$ or $T \vdash \neg\varphi$.

Theorem 4.6 (Łoś–Vaught test). If T has no finite models and is κ -categorical, for some infinite cardinal κ bigger or equal the cardinality of L , then T is complete.

Proof. Assume for a contradiction that there is a sentence φ such that $T \not\vdash \varphi$ and $T \not\vdash \neg\varphi$. By completeness (and deduction), there are $M \models T \cup \{\neg\varphi\}$ and $N \models T \cup \{\varphi\}$ of the cardinality of L . By the (upper) Löwenheim–Skolem theorem we can find elementary extensions $M \prec M', N \prec N'$ of cardinality κ . Using κ -categoricity we get that $M' \cong N'$. Hence

$$M \equiv M' \equiv N' \equiv N$$

and we get a contradiction. □

Example 4.7. Consider the theory DLO_0 of dense linear orders without endpoints. It is a theory in the language $L = \{<\}$ axiomatized by the usual linear order axioms together with

$$(\exists x)(\exists y)(x < y) \wedge (\forall x, y)(x < y \rightarrow (\exists w, u, z)(w < x < u < y < z)).$$

Consider two countable models $M, N \models \text{DLO}_0$. We will show that $M \cong N$ (Cantor's theorem). Enumerate $M = \{a_n : n \in \mathbb{N}\}, N = \{b_n : n \in \mathbb{N}\}$. We construct inductively a sequence of partial isomorphisms f_n , where $A_n \subseteq M, B_n \subseteq N$ are sequences of finite subsets such that

- i. $A_n \subseteq A_{n+1}, B_n \subseteq B_{n+1}, f_n \subseteq f_{n+1}, A_0 = B_0 = f_0 = \emptyset$,
- ii. $f_n : A_n \rightarrow B_n$ is an isomorphism of induced finite orders,
- iii. $a_n \in A_{2n+1}, b_n \in B_{2n+2}$.

The construction is inductive. Assume we have f_{2n}, A_{2n}, B_{2n} . If $a_n \in A_{2n}$, we put $(f_{2n+1}, A_{2n+1}, B_{2n+1}) = (f_{2n}, A_{2n}, B_{2n})$. Otherwise we write elements of A_{2n} as

$$c_1 < \cdots < c_{i-1} < a_n < c_i < \cdots < c_k,$$

and we find $b \in N$ such that $f_{2n}(c_{i-1}) < b < f_{2n}(c_i)$ (if a_n is the biggest or the smallest element we choose b appropriately). Then we put $f_{2n+1} = f_{2n} \cup \{(a_n, b)\}$ and $A_{2n+1} = A_{2n} \cup \{a_n\}, B_{2n+1} = B_{2n} \cup \{b\}$. On the odd steps we reverse the construction. If we put $f = \bigcup_{n \in \mathbb{N}} f_n$ we get an isomorphism $f : M \rightarrow N$. In particular DLO_0 is complete.

Corollary 4.8. DLO_0 is decidable. More precisely, there is an algorithm that produces sentences true in any model of DLO_0 , such that for any sentence σ , it will produce σ or $\neg\sigma$ at some point.

Proof. Note that DLO_0 is given by a list of finitely many axioms. If our program lists all formal proofs that use this list of axioms, by completeness of DLO_0 we know that it has the desired property. $\square$

In the above proof, it would be enough to know that DLO_0 is recursively axiomatizable, i.e., that there exists a computer program listing its axioms (and possibly working till the end of time).

Another tool in determining elementary equivalence of L -structures is quantifier-elimination.

Definition 4.9. We say that an L -theory T *eliminates quantifiers* if for every L -formula φ there is a quantifier-free formula ψ such that $T \vdash \varphi \leftrightarrow \psi$.

We start with an example. Let $L = \{E\}$ where E is a binary relation symbol and let T be the theory saying that E is an equivalence relation with infinitely many infinite classes. We will prove directly that it has quantifier elimination. Consider a quantifier-free formula

$$\psi(\bar{x}, y) = \bigvee_{i=1}^n \bigwedge_{j=1}^{J_i} (\alpha_{ij} F_{ij} \beta_{ij}),$$

where $\alpha_{ij}, \beta_{ij} \in \{x_1, \dots, x_n, y\}$ and $F_{ij} \in \{=, \neq, E, \neg E\}$. We will show that $\exists y \psi(\bar{x}, y)$ is T -equivalent to a quantifier-free formula in variables $\bar{x}$. Without loss of generality we can assume that

$$\psi(\bar{x}, y) = \bigwedge_{j=1}^m (\alpha_j F_j \beta_j).$$

Split this conjunction into terms that do not contain y and the other ones, so that we can assume that, up to rearranging and using the symmetry of relations considered, we have

$$\psi(\bar{x}, y) = \bigwedge_{j=1}^k (x_{i_j} F_j x_{i'_j}) \wedge \bigwedge_{l=k+1}^m (y F_l x_{i_l}).$$

If one of the relations $y F_l x_{i_l}$ is $y = x_{i_l}$, then $\exists y \psi(\bar{x}, y)$ is equivalent to $\psi(\bar{x}, x_{i_l})$. Otherwise, assume that the new sentences say that

$$y \neq x_{j_1}, \dots, y \neq x_{j_a}, y E x_{j_{a+1}}, \dots, y E x_{j_b}, \neg y E x_{j_{b+1}}, \dots, \neg y E x_{j_p}.$$

If $a = b$, then in a model of T , $\exists y \psi(\bar{x}, y)$ holds for a tuple $\bar{c}$ if and only if $\bar{c}$ satisfies

$$\bigwedge_{j=1}^k (x_{i_j} F_j x_{i'_j}).$$

If $a < b$, then in a model of T , $\exists y \psi(\bar{x}, y)$ holds for a tuple $\bar{c}$ if and only if $\bar{c}$ satisfies

$$\bigwedge_{j=1}^k (x_{i_j} F_j x_{i'_j}) \wedge \bigwedge_{l=a+1}^{b-1} (x_{j_l} E x_{j_b}) \wedge \bigwedge_{l=b+1}^p (\neg x_{j_b} E x_{j_l}).$$

Thus we showed that T has quantifier-elimination. Note that the same algorithm works, if we added two constants to L , say $L' = L \cup \{c, d\}$. However, with two constants the theory $T' = T$ (but in L') is not complete, as it does not determine whether $c E d$ or whether $c = d$. Quantifier elimination for T' implies that every L' -sentence is T' -equivalent to a Boolean combination of sentences $c E d, c = d$. Thus determining these relations is the only obstruction to completeness of T' . Later we will develop more conceptual criteria for quantifier-elimination. To do that we need to introduce types.

5. STONE DUALITY AND ULTRAPRODUCTS

Fix a theory T in L . We say that two formulas $\varphi(\bar{x}), \psi(\bar{x})$ are *equivalent* (modulo T) if $T \vdash \varphi \leftrightarrow \psi$. Equivalence classes of formulas form a *Boolean algebra*, i.e., we can define operations $\wedge, \vee, \neg$ on them, and the usual laws of set operations $\cap, \cup, (-)^c$ are satisfied ($\wedge, \vee$ are associative, idempotent, commutative, and distributive with respect to each other; $\neg$ is involutive, satisfies the law of excluded middle, and De Morgan's laws hold). Furthermore, this algebra has $0 = [\bigwedge_i x_i \neq x_i]$ and $1 = [\bigwedge_i x_i = x_i]$.

Definition 5.1. Fix a tuple of variables $\bar{x} = (x_1, \dots, x_n)$. The *Lindenbaum-Tarski algebra* is the Boolean algebra of T -equivalence classes of formulas in variables $\bar{x}$, where we define $[\varphi] \wedge [\psi] = [\varphi \wedge \psi]$, etc.

Definition 5.2. A *type* in variables $\bar{x} = (x_1, \dots, x_n)$ is a collection of formulas

$$p(\bar{x}) = \{\varphi_i(\bar{x}) : i \in I\}$$

such that every finite subset $p_0(\bar{x}) \subseteq p(\bar{x})$, $T \cup \{\exists \bar{x} \bigwedge p_0(\bar{x})\}$ is consistent. We call p a *complete type*, if for any formula φ in variables $\bar{x}$, either $\varphi \in p$ or $\neg \varphi \in p$.

Non-complete (or not necessarily complete) types are sometimes called *partial types*.

Exercise 5.3. (e) Let $p(\bar{x})$ be a type with respect to T . Show that there is a model $M \models T$ and a tuple $\bar{a} \in M^{|\bar{x}|}$ such that for every $\psi \in p$ we have $M \models \psi(\bar{a})$.

Let $(B, \wedge, \vee, \neg, 0, 1)$ be a Boolean algebra. A (proper) *filter* is a subset $F \subseteq B$ such that ($0 \notin F$ and):

- $1 \in F$,
- $a \in F \ \& \ \neg a \vee b = 1 \implies b \in F$,
- $a, a' \in F \implies a \wedge a' \in F$.

In the second bullet the relation $\neg a \vee b = 1$ is sometimes denoted by $a \subset b$.

Exercise 5.4. (e) Define $a \triangle b = (a \wedge \neg b) \vee (b \wedge \neg a)$. Show that B with $+$ $:= \triangle$, $\cdot$ $:= \wedge$ is a commutative ring with $0, 1$ being respectively the neutral elements of addition and multiplication. Show that an $F \subseteq B$ is a proper filter if and only if $\{\neg b : b \in F\}$ is a proper ideal.

A proper filter is called an *ultrafilter* if it is maximal with respect to inclusion of proper filters. In particular, $F \subseteq B$ is an ultrafilter if and only if $\{\neg b : b \in F\}$ is a maximal proper ideal in $(B, +, \cdot)$. A proper ideal in $(B, +, \cdot)$ is maximal if and only if it is prime; the quotient by such an ideal is the two-element field (e).

Exercise 5.5. (e) A filter $F \subseteq B$ is an ultrafilter if and only if for every $b \in B$, exactly one of the two possibilities happen: $b \in F$ or $\neg b \in F$.

Definition 5.6. The *Stone space* $\text{Stone}(B)$ of a Boolean algebra B is the set of ultrafilters on B equipped with the topology where the basis open sets are (finite intersections of) sets of the form

$$[b] := \{F \in \text{Stone}(B) : b \in F\},$$

where we vary $b \in B$. Equivalently, we could define $\text{Stone}(B)$ as the set of prime ideals in $(B, +, \cdot)$, equipped with the Zariski topology ((e) for those who know some algebraic geometry).

For a tuple of variables $\bar{x} = (x_1, \dots, x_n)$ we denote by $S_n(T)$ (or $S_{\bar{x}}(T)$) the Stone space of the Lindenbaum-Tarski algebra in $\bar{x}$. We call it the space of *complete n -types*.

Note that we do not need to consider finite intersections of sets of the form $[b]$ as $[b_1] \cap [b_2] = [b_1 \wedge b_2]$ (e).

Formally, an element of $S_n(T)$ is an ultrafilter of T -equivalence classes of L -formulas in variables $x_1, \dots, x_n$. We will be identifying such an element with the corresponding complete type consisting of all representatives of these equivalence classes.

Proposition 5.7. Stone space of any Boolean algebra B is compact, Hausdorff, and has a basis consisting of clopen sets.

Proof. First note that $[b]_{b \in B}$ is a basis of clopen subsets as $[b]^c = [\neg b]$. If $F_1, F_2 \subseteq B$ are two different ultrafilters in B , then there is $b \in B$ such that $b \in F_1$ and $\neg b \in F_2$, so that $F_1 \in [b]$ & $F_2 \in [b]^c = [\neg b]$. This shows Hausdorffness.

For compactness, assume that $[b_i]_{i \in I}$ is a centered family. This means that for every finite $I_0 \subseteq I$ the set $\bigcap_{i \in I_0} [b_i]$ is non-empty, i.e., the ideal generated by $(\neg b_i : i \in I_0)$ in $(B, +, \cdot)$ is a proper ideal. Thus, the ideal generated by all $\neg b_i$'s is a proper ideal, and any maximal ideal containing it will witness $\bigcap_{i \in I} [b_i] \neq \emptyset$. $\square$

A topological space satisfying the conditions from the preceding proposition is called a *Stone space*. The set of clopen subsets of a Stone space X is a Boolean algebra $\text{Bool}(X)$.

Theorem 5.8 (Stone duality). There is a correspondence between Boolean algebras and Stone spaces. More precisely, $\text{Bool}(\text{Stone}(B)) \cong B$, $\text{Stone}(\text{Bool}(X)) \approx X$ for any Boolean algebra B and any Stone space X .

Proof. Exercise ⑥. □

For those who already know category theory (we will talk about it soon) - Stone duality is an (anti-)equivalence of categories when we consider continuous maps between Stone spaces and homomorphisms between Boolean rings.

If I is a set and $\mathcal{U} \subseteq P(I)$ is an ultrafilter on the Boolean algebra of all subsets of I , we can identify $\mathcal{U}$ with a finitely additive probabilistic measure $\mu : P(I) \rightarrow \{0, 1\}$, where $\mu(A) = 1 \iff A \in \mathcal{U}$.

Definition 5.9. Let $(M_i)_{i \in I}$ be a collection of L -structures and let $\mathcal{U}$ be an ultrafilter on $P(I)$. The *ultraproduct of $(M_i)_{i \in I}$ with respect to $\mathcal{U}$* , denoted $\prod_{\mathcal{U}} M_i$ or $\prod_{i \in I} M_i / \mathcal{U}$, is the following L -structure:

- (i) The universe of $\prod_{i \in I} M_i / \mathcal{U}$ is the set of equivalence classes of sequences in $\prod_{i \in I} M_i$ where we identify two sequence that are equal ‘almost everywhere’ with respect to the measure μ . We denote the class of $(a_i) \in \prod_{i \in I} M_i$ by $[(a_i)]_{\mathcal{U}}$.
- (ii) If f is a function symbol of arity n , then we define its interpretation in $\prod_{i \in I} M_i / \mathcal{U}$ to be:

$$f([(a_i^1)]_{\mathcal{U}}, \dots, [(a_i^n)]_{\mathcal{U}}) = [(f(a_i^1, \dots, a_i^n))]_{\mathcal{U}}.$$

- (iii) If P is a predicate symbol of arity m , then we define its interpretation in $\prod_{i \in I} M_i / \mathcal{U}$ to be:

$$P([(b_i^1)]_{\mathcal{U}}, \dots, [(b_i^m)]_{\mathcal{U}}) \iff \{i \in I : M_i \models P(b_i^1, \dots, b_i^m)\} \in \mathcal{U},$$

- (iv) If c is a constant symbol, then we define its interpretation in $\prod_{i \in I} M_i / \mathcal{U}$ to be $[(c^{M_i})]_{\mathcal{U}}$.

We leave it to the reader ⑥ to verify that in this definition of L -structure function and predicate symbols interpretations only depend on elements of $\prod_{i \in I} M_i / \mathcal{U}$ and not of their representatives in $\prod_{i \in I} M_i$.

Theorem 5.10 (Łoś’s theorem). Let $\varphi(x_1, \dots, x_n)$ be an L -formula. Then

$$\prod_{i \in I} M_i / \mathcal{U} \models \varphi([(a_i^1)]_{\mathcal{U}}, \dots, [(a_i^n)]_{\mathcal{U}}) \iff \{i \in I : M_i \models \varphi(a_i^1, \dots, a_i^n)\} \in \mathcal{U}.$$

Proof. We do it inductively based on the complexity of φ . The case of atomic formulas and connectives is left as an exercise ⑥. Let $\varphi(\bar{x}) = \exists y \psi(\bar{x}, y)$. First, assume that $I_0 = \{i \in I : M_i \models \varphi(a_i^1, \dots, a_i^n)\} \in \mathcal{U}$. This means that for $i \in I_0$ we can find $b_i \in M_i$ such that $M_i \models \psi(a_i^1, \dots, a_i^n, b_i)$. Define $(b_i) \in \prod_{i \in I} M_i$ arbitrarily outside of I_0 . By the inductive hypothesis we have

$$\prod_{i \in I} M_i / \mathcal{U} \models \psi([(a_i^1)]_{\mathcal{U}}, \dots, [(a_i^n)]_{\mathcal{U}}, [(b_i)]_{\mathcal{U}}),$$

so by definition the left hand side of the desired equivalence holds.

Now, let us assume that

$$\prod_{i \in I} M_i / \mathcal{U} \models \varphi([(a_i^1)]_{\mathcal{U}}, \dots, [(a_i^n)]_{\mathcal{U}}).$$

Let $[(b_i)]_{\mathcal{U}}$ be such that $\prod_{i \in I} M_i / \mathcal{U} \models \psi([(a_i^1)]_{\mathcal{U}}, \dots, [(a_i^n)]_{\mathcal{U}}, [(b_i)]_{\mathcal{U}})$. By the inductive hypothesis, this means that $\{i \in I : M_i \models \psi(a_i^1, \dots, a_i^n, b_i)\} \in \mathcal{U}$. However this set is contained in $\{i \in I : M_i \models \exists y \psi(a_i^1, \dots, a_i^n, y)\}$, so we get that the latter also is in $\mathcal{U}$. $\square$

An ultrafilter $\mathcal{U} \subseteq P(I)$ is called *principal*, if it is of the form

$$\mathcal{U} = \{J \subseteq I : i_0 \in J\}$$

for some $i_0 \in I$. In particular, an ultrafilter is non-principal if and only if it contains the filter of all cofinite subsets of I (©). Non-principal ultrafilter are very useful as the ultraproducts with respect to them give a way to ‘average’ an infinite family of structures into a single one.

6. REALISING AND OMITTING A TYPE

Fix an L -theory T .

Definition 6.1. Let $A \subseteq M$ be a subset of $M \models T$. We write $L(A) = L \cup \{c_a : a \in A\}$ for the expansion of the language by copies of elements of A . For $\bar{b} \in M^{|\bar{x}|}$ we write $\text{tp}(\bar{b}/A)$ for the set of equivalence classes of formulas $\varphi(\bar{x}) \in L(A)$ such that $M \models \varphi(\bar{b})$. We denote by $S_{\bar{x}}(A)$ the space of complete $\bar{x}$ -types relative to $\text{Diag}_{\text{el}}(A)$ as an $L(A)$ -theory. In particular $\text{tp}(\bar{b}/A) \in S_{\bar{x}}(A)$.

Note that a type $p \in S_{\bar{x}}(A)$ carries information about the complete theory $\text{Th}(M)$ (even $\text{Diag}_{\text{el}}(A)$) that extends T . Indeed, this is because sentences can be treated as formulas in variables $\bar{x}$ (e.g. by conjugating $\sigma \wedge \bigwedge_{i=1}^n (x_i = x_i)$ for an $L(A)$ -sentence σ). For this reason, type spaces over parameter sets are typically considered when one is interested in a complete theory (or in a specific structure and its elementary extensions). In that case, when a complete theory T is clear from the context, we write $S_n(\emptyset)$ for $S_n(T)$.

Lemma 6.2. Consider a type $p \in S_n(A)$. There is an elementary extension $M \prec N$ and $\bar{b} \in N^n$ such that $p = \text{tp}(\bar{b}/A)$.

Proof. Consider the theory $T' = \text{Diag}_{\text{el}}(M) \cup p(\bar{x})$ in the language $L' = L(M) \cup \{x_1, \dots, x_n\}$ (so that we now treat m 's from M and x_i 's as constants). Choose a finite $p_0 \subseteq p$. Then, by definition

$$(m)_{m \in M} \models \text{Diag}_{\text{el}}(M) \text{ and } M \models \exists \bar{x} \bigwedge p_0(\bar{x}),$$

so if $\bar{c} \in M^n$ is a witness for the latter, then M with the natural $L(M)$ -structure, and with $x_i^M = c_i$ shows that $\text{Diag}_{\text{el}}(M) \cup p_0(\bar{x})$ is consistent. Since p_0 was arbitrary finite subset of p , we are done by compactness. $\square$

Since now on we allow ourselves to be a bit sloppy about extending the language by constants as in the above proof, in the sense that sometimes we will be considering variables or elements of a model as new constant symbols. In particular, we will see a type $p \in S_n(A)$ as a collection of formulas

$$p(\bar{x}) = \{\varphi_i(\bar{x}, \bar{a}_i) : i \in I\},$$

such that for every L -formula $\psi(\bar{x}, \bar{y})$ and $\bar{a} \in A^{|\bar{y}|}$ either $\psi(\bar{x}, \bar{a}) \in p$ or $\neg\psi(\bar{x}, \bar{a}) \in p$.

Definition 6.3. A partial type $p(\bar{x})$ is *isolated*, if there is a formula $\varphi(\bar{x})$ such that $T \cup \{\exists \bar{x} \varphi(\bar{x})\}$ is consistent and for every $\psi \in p$ we have

$$T \vdash \forall \bar{x} (\varphi(\bar{x}) \rightarrow \psi(\bar{x})).$$

A type $p(\bar{x})$ is *realised in a model* $M \models T$ if there is some tuple $\bar{a} \in M^n$ such that for every $\psi \in p$ we have $M \models \psi(\bar{a})$. We denote it by $\bar{a} \models p$, if M is clear from the context.

In the context of the above definition we say that $\varphi(\bar{x})$ isolates $p(\bar{x})$. In a complete theory, an isolated type is realised in every model $\textcircled{e}$. This observation can be made into a characterisation.

Theorem 6.4 (Omitting types theorem). Let $p(\bar{x})$ be a non-isolated partial type with respect to a theory T in a countable language L . Then there is a model $M \models T$ that does not realise p (so M omits p).

Proof. We modify Henkin's proof of completeness. Let $L' = L \cup \{d_n : n \in \mathbb{N}\}$. Let $(\varphi_i(z))_{i \in \mathbb{N}}$ be a sequence of all L' -formulas with one free variable z . Fix an enumeration of all $|\bar{x}|$ -tuples of new constants $(\bar{d}_1, \bar{d}_2, \dots)$. Let $g : \mathbb{N} \rightarrow \mathbb{N}$ be an increasing function such that for every $l \in \mathbb{N}$, $d_{g(l)}$ does not appear in $\varphi_i(\bar{x}), \bar{d}_i$ for $i \leq l$. Now, we construct an increasing sequence of L' -theories $(T_i)_{i \in \mathbb{N}}$, each a finite extension of T , in the following manner. First, $T_0 = T$. Next, for $j \in \mathbb{N}$ such that T_j is defined, we construct T_{j+1} in two steps:

- $T'_j = T_j \cup \{\neg\psi(\bar{d}_j)\}$ for some formula $\psi \in p$. We claim that we can choose ψ so that T'_j is consistent. Indeed, if this was not the case, then for all $\psi \in p$, we have $T_j \vdash \psi(\bar{d}_j)$. Hence, if we denote by $(\bar{c}, \bar{d}_j)$ all new constants appearing in T_j and by $\chi(\bar{c}, \bar{d}_j)$ the conjunction of all new added formulas, we get that

$$T \vdash \chi(\bar{c}, \bar{d}_j) \rightarrow \psi(\bar{d}_j)$$

for all $\psi \in p$. Because we are only using new constants, we infer that

$$T \vdash \exists \bar{y} \chi(\bar{y}, \bar{x}) \rightarrow \psi(\bar{x}).$$

This gives a contradiction with the fact that p is not isolated.

- We put $T_{j+1} = T'_j \cup \{\exists z \varphi_j(z) \rightarrow \varphi_j(d_{g(j)})\}$. The proof that T_{j+1} is consistent is the same as in the proof of Theorem 2.15.

We now let $T_\infty = \bigcup_{i \in \mathbb{N}} T_i$ and we choose a completion T' of T_∞ . We construct a model from equivalence classes of constants $(d_n)_{n \in \mathbb{N}}$ in the same way as before. By the construction, this model will satisfy the axioms from T and will omit p $\textcircled{e}$. $\square$

This result is not valid for uncountable theories. Indeed consider the language $L = \{c_\lambda : \lambda < \aleph_1\} \cup \{d_n : n < \omega\}$ and the L -theory

$$T = \{c_\alpha \neq c_\beta : \alpha \neq \beta < \aleph_1\}.$$

Let $p(x) = \{x \neq d_n : n < \omega\}$. Then every model of T is uncountable, in particular it realises p . However p is not isolated $\textcircled{e}$.

Another result using isolated types is the following characterisation of $\aleph_0$ -categorical theories (often called ω -categorical).

Theorem 6.5 (Engeler, Ryll-Nardzewski, Svenonius, independently in 1959). Let T be a complete theory in a countable language with infinite models. Then T is ω -categorical if and only if for every n the space $S_n(T)$ is finite.

Proof. First, assume that for some n , $S_n(T)$ is not finite. Then there is $p \in S_n(T)$ that is not isolated ⑥. Let M be a countable model of T that realises p and let N be a countable model omitting p . Then M and N cannot be isomorphic ⑥.

On the other hand, assume that $S_n(T)$ is finite for every $n \in \mathbb{N}$. Let M and N be two countable models of T . We build an isomorphism between them using the back-and-forth method as in Example 4.7. Assume we have chosen $\bar{a} \in M^n$ and $\bar{b} \in N^n$ such that they satisfy the same n -type in $S_n(T)$. Consider a new element $c \in M$. We will find $d \in N$ such that $\bar{a}c$ and $\bar{b}d$ (concatenated tuples) satisfy the same $(n+1)$ -types in $S_{n+1}(T)$. Let $\psi(\bar{x}, y)$ be the formula isolating $\text{tp}(\bar{a}, c) \in S_{n+1}(T)$ and let $\xi(\bar{x})$ be the formula isolating $\text{tp}(\bar{a}) \in S_n(T)$. We claim that $M \models \forall \bar{x} (\xi(\bar{x}) \rightarrow \exists y \psi(\bar{x}, y))$. This is because $\exists y \psi(\bar{x}, y) \in \text{tp}(\bar{a})(\bar{x})$ and ξ isolates $\text{tp}(\bar{a}) = \text{tp}(\bar{b})$. Since T is complete, the same sentence holds in N , so $N \models \exists y \psi(\bar{b}, y)$. Any $d \in N$ satisfying $N \models \psi(\bar{b}, d)$ will have the desired property. $\square$

The topology of the space of types can be used to deduce criteria for quantifier elimination.

Proposition 6.6. A formula $\varphi(\bar{x})$ is T -equivalent to a quantifier-free formula if and only if for every L -structure A and any two models $A \subseteq M, N \models T$ containing A , for any $\bar{a} \in A^{|\bar{x}|}$, we have

$$M \models \varphi(\bar{a}) \iff N \models \varphi(\bar{a}).$$

Proof. We only prove the implication from right to left. Let $n = |\bar{x}|$ and write $S_n^{\text{qf}}(T)$ for the Stone space associated to the Boolean algebra of T -equivalence classes of quantifier-free formulas in L . The embedding of this Boolean algebra into the Lindenbaum-Tarski algebra in n -variables induces a continuous surjection

$$\pi : S_n(T) \rightarrow S_n^{\text{qf}}(T).$$

Let $D = \pi[\varphi], E = \pi[\neg\varphi]$. First we show that $D \cap E = \emptyset$. Indeed, otherwise there would be a point $p_0 \in S_n^{\text{qf}}(T)$ that has two extensions $p_1 \in [\varphi] \subseteq S_n(T)$ and $p_2 \in [\neg\varphi] \subseteq S_n(T)$. Let $(M_1, \bar{a}_1) \models p_1$ and $(M_2, \bar{a}_2) \models p_2$. Since the quantifier-free type of $\bar{a}_1$ and $\bar{a}_2$ are the same, they span isomorphic L -substructures $A_1 \subseteq M_1$ and $A_2 \subseteq M_2$ respectively ⑥. By identifying $A_1 = A_2 =: A$ and $\bar{a}_1 = \bar{a}_2 =: \bar{a}$, we find a configuration as in the assumption of the proposition, but by construction $M_1 \models \varphi(\bar{a})$ and $M_2 \models \neg\varphi(\bar{a})$. Thus indeed $D \cap E = \emptyset$ and since both are images of compact spaces by a continuous map, they are both closed, hence clopen. This finishes the proof by one of the exercises below. $\square$

Exercises, sheet 3.

1. Prove the Ehrenfeucht–Fraïssé theorem.
2. Prove that the following $L = \{E\}$ -theories have quantifier elimination:
 - E is an equivalence relation with infinitely many equivalence classes of cardinality 7,
 - E is an equivalence relation with 7 infinite equivalence classes.
3. Solve all small exercises about Boolean algebras.

4. Prove that a subset $X \subseteq S_n(T)$ is clopen if and only if it is of the form $[\varphi]$ for some formula φ in n variables. Characterise closed subsets of $S_n(T)$. What about $\text{Stone}(B)$ for a Boolean algebra B ?
5. Prove that a complete type $p \in S_n(T)$ is isolated if and only if $\{p\}$ is a clopen set in $S_n(T)$.
6. Use ultraproducts to give another proof of the compactness theorem (this proof generalises easily to continuous logic).
7. Let $p(\bar{x})$ be a countable collection of formulas in variables $\bar{x}$ and assume that M_i are L -structures such that p is a type with respect to $\text{Th}(M_i)$, for $i \in \mathbb{N}$. Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$. Show that p is realised in $\prod_{i \in \mathbb{N}} M_i / \mathcal{U}$.
8. Prove that $|S_n(A)| \leq 2^{|A|+|L|}$.

The following is an optional exercise for fans of descriptive set theory. Let T be a countable theory in a countable language L . Consider

$$X = \prod_f \omega^{\omega^{n_f}} \times \prod_P 2^{\omega^{m_P}} \times \prod_c \omega,$$

where the products are taken over the sets of function, predicate and constant symbols in L respectively. Let $\text{Mod}_\omega(T) \subseteq X$ be the subset consisting of L -structures on ω that are models of T . Define a topology on X with basic open subsets of the form

$$\{M \in X : M \models \varphi(\bar{a})\},$$

where $\varphi(\bar{x})$ varies over L -formulas and $\bar{a} \in \omega^{|\bar{x}|}$. Prove that:

- X is a Polish space.
- $\text{Mod}_\omega(T) \subseteq X$ is a closed subset, hence also a Polish space.
- If a partial type p is not isolated, then the set $\{M \in \text{Mod}_\omega(T) : M \text{ omits } p\}$ is a dense G_δ in $\text{Mod}_\omega(T)$.
- For a countable collection of non-isolated types $p_1, p_2, p_3, \dots$, there is a countable model $M \models T$ that omits all of them.

7. SATURATED MODELS

Let T be a **complete** L -theory and let κ be an infinite cardinal. Let $M, N \models T$ and let $A \subseteq M, B \subseteq N$ be subsets.

Definition 7.1. We call a function $f : A \rightarrow B$ *elementary* if for all $\varphi(\bar{x})$ and all $\bar{a} \in A^{|\bar{x}|}$ we have

$$M \models \varphi(\bar{a}) \iff N \models \varphi(f(\bar{a})),$$

where we extend f coordinate-wise.

We write $L(A)$ for the language L extended by constants from A . By a type over A we mean a type in the language $L(A)$ (with respect to the elementary diagram of A). Note that for subsets $A \subseteq B \subseteq M$ we have a continuous restriction map $S_n(B) \rightarrow S_n(A)$. If $p(\bar{x})$ is a type over M or over a subset of M , we write $p(M) \subseteq M^{|\bar{x}|}$ for the set of its realisations.

Definition 7.2. We call $M \models T$:

- (1) κ -*saturated* if for all subsets $A \subseteq M$ with $|A| < \kappa$ and for all $p \in S_1(A)$, we have $p(M) \neq \emptyset$. We call M *saturated* if it is $|M|$ -saturated.

- (2) κ -universal if for all $N \models T$ with $|N| < \kappa$, there is an elementary embedding $f : N \rightarrow M$. M is *universal* if it is $|M|$ -universal.
- (3) κ -homogeneous if for all subsets $A \subseteq M$ with $|A| < \kappa$ and $a \in M$ every elementary function $f : A \rightarrow M$ can be extended to an elementary function $g : A \cup \{a\} \rightarrow M$. M is *homogeneous* if it is $|M|$ -homogeneous.
- (4) *strongly* κ -homogeneous if for all subsets $A \subseteq M$ with $|A| < \kappa$, every elementary function $f : A \rightarrow M$ extends to an automorphism of M . M is *strongly homogeneous* if it is M -strongly homogeneous.
- (5) κ -compact if for any partial 1-type $p(x)$ over M with $|p(x)| < \kappa$ we have $p(M) \neq \emptyset$.
- (6) κ -monster model if it is κ -saturated and strongly κ -homogeneous.
- (7) $(< \aleph_0)$ -universal if for all natural n and for all $p \in S_n(\emptyset)$, $p(M) \neq \emptyset$.

Proposition 7.3. The following implications hold.

- i) If M is κ -saturated, then it is also κ -homogeneous and κ -universal.
- ii) If M is κ -homogeneous and $(< \aleph_0)$ -universal, then it is κ -saturated.

Proof. First we prove i). Fix a subset $A \subseteq M$ of cardinality smaller than κ , $a \in M$, and let $f : A \rightarrow M$ be an elementary function. We want to extend it to an elementary function $g : A \cup \{a\} \rightarrow M$. Let $p = \text{tp}(a/A)$. Denote by $q = f(p)$ the image of p by f , which can be defined as follows:

$$f(p) = \{\varphi(x, f(\bar{b})) : \varphi(x, \bar{b}) \in p(x)\} \in S_1(f(A)).$$

Since M is κ -saturated, there is $c \in M$ such that $\text{tp}(c/f(A)) = q$. We claim that g extended by $g(a) = c$ is elementary. Indeed, if $M \models \varphi(a, \bar{b})$, then $\varphi(x, \bar{b}) \in p(x)$, so $\varphi(x, f(\bar{b})) \in q(x)$ and $M \models \varphi(c, f(\bar{b}))$, i.e., $M \models \varphi(f(a), f(\bar{b}))$. To prove κ -universality, fix a model $N \models T$ of size smaller than κ . Enumerate

$$N = \{a_\alpha : \alpha < \lambda\}$$

for some cardinal $\lambda < \kappa$. Let $B_\beta = \{a_\alpha : \alpha < \beta\}$ for $\beta \leq \lambda$. We construct inductively an increasing sequence of elementary maps $f_\beta : B_\beta \rightarrow M$ (in the sense of inclusion) starting from $f_0 = \emptyset$. Assume we have defined f_γ for all $\gamma < \beta$. Let $f = \bigcup_{\gamma < \beta} f_\gamma$ and let $A = \bigcup_{\gamma < \beta} f_\gamma(B_\gamma) \subseteq M$ be its image. Denote by p the type of a_β over B_β and let $q = f(p) \in S_1(A)$. Since $|A| < \kappa$, there is $c \in M$ such that $\text{tp}(c/A) = q$. A similar argument as before shows that $f_\beta := f \cup \{(a_\beta, c)\}$ is an elementary function $f_\beta : B_\beta \rightarrow M$. For $\beta = \lambda$ we get an elementary embedding $N \rightarrow M$.

Now we prove ii). Fix a type $p \in S_1(A)$ for a subset $A \subseteq M$ with $|A| < \kappa$. We will show inductively based on the size of A that $p(M) \neq \emptyset$. First, let us assume that A is finite. Let $M \prec N$ be an elementary extension together with $b \in N$ such that $\text{tp}(b/A) = p$. Write $A = \{a_1, \dots, a_n\}$ and let $q = \text{tp}(a_1, \dots, a_n, b/\emptyset)$. Fix a realisation $(a'_1, \dots, a'_n, b')$ of q in M . Consider the function f that sends a'_i to a_i for $i = 1, \dots, n$. It is elementary because

$$\text{tp}(a'_1, \dots, a'_n/\emptyset) = \text{tp}(a_1, \dots, a_n/\emptyset).$$

Hence, by κ -universality, it extends to an elementary function $f' : \{a'_1, \dots, a'_n\} \cup \{b'\} \rightarrow M$. If $c = f'(b')$, then $\text{tp}(c/a_1, \dots, a_n) = f'(\text{tp}(b'/a'_1, \dots, a'_n)) = p$, where the last equality follows from the fact that if $g : A \cup \{b\} \rightarrow M$ is an elementary function, then $g(\text{tp}(b/A)) = \text{tp}(g(b)/g(A))$ ⑥.

This proves the case of finite A and we can pass to the general case. Assume that $|A| = \mu < \kappa$, enumerate $A = \{a_\alpha : \alpha < \mu\}$ and fix $p \in S_1(A)$. Recall that by the inductive hypothesis we can assume that M is λ -saturated for every $\lambda < \mu$. Let b be a realisation of p in an elementary extension $M \prec N$ and let $b' \in M$ be the realisation of $p|_\emptyset$ (the restriction of p to the empty set of parameters). We will construct $A' = \{a'_\alpha : \alpha < \mu\} \subseteq M$ such that $f : A \cup \{b\} \rightarrow A' \cup \{b'\}$ mapping a_α to a'_α and b to b' is elementary. Assume that we have constructed a'_α for $\alpha < \gamma$ with some $\gamma < \mu$, such that the function

$$g : a_\alpha \mapsto a'_\alpha, b \mapsto b', \text{ for } \alpha < \gamma,$$

is elementary. Let $r = \text{tp}(a_\gamma/b, a_{\alpha < \gamma})$ and put $q = g(r) \in S_1(\{b'\} \cup \{a'_\alpha : \alpha < \gamma\})$. By the inductive hypothesis, there is $a'_\gamma \models q$ in M . Together we obtain the desired function f . Now, by the definition of κ -homogeneity for $A' \subseteq M$, $b' \in M$, and the function $f^{-1}|_{A'} : A' \rightarrow A$, we get that there is $c \in M$ such that $f^{-1}|_{A' \cup \{(b', c)\}}$ is elementary. Then

$$\text{tp}(c/A) = f^{-1}(\text{tp}(b'/A')) = \text{tp}(b/A) = p$$

and we are done. $\square$

Remark 7.4. Note that for a countable T , if $M \models T$ is κ -universal, then it is $(< \aleph_0)$ -universal. Indeed, for a type $p \in S_1(\emptyset)$, if N is a countable model where $b \models p$, then for an embedding $j : N \rightarrow M$ we have $j(b) \models p$.

Theorem 7.5. Let $M, N \models T$ be saturated of the same cardinality. Then $M \cong N$.

Proof. We use the back-and-forth method. Assume that $|M| = |N| = \kappa$ and that we already have a partial elementary bijection $f : A \rightarrow B$ for some $A \subseteq M, B \subseteq N$ with $|A| = |B| < \kappa$. For $a \in M$ we will find $b \in N$ such that $g = f \cup \{(a, b)\}$ is elementary. Let $p = \text{tp}(a/A)$ and define $q = f(p)$. Since N is saturated and $|B| < \kappa = |M|$, there is a $b \in N$ such that $\text{tp}(b/B) = q$. This b will satisfy our constraint, and by going back-and-forth we are done. $\square$

From the proof we also get the following.

Corollary 7.6. Let M be a saturated model of T . Then M is strongly homogeneous.

Under sufficiently strong assumptions on a cardinal κ , saturated models of size κ exist (and if they do, they are unique up to an isomorphism). For simplicity, in the following we assume that the language L is countable.

Theorem 7.7. Assume that $\kappa = 2^{<\kappa}$, $\text{cf}(\kappa) = \kappa$, $\kappa > \aleph_0$. Then there exists a saturated $M \models T$ of size κ .

Proof. For a model $N \models T$ with $|N| \leq \kappa$ we define

$$X_N = \bigcup \{S_1(A) : A \subseteq N, |A| < \kappa\},$$

where A varies over subsets. Note that $|X_N| \leq \kappa$ because for each $A \subseteq N$, $|S_1(A)| \leq 2^{|A| + \aleph_0} \leq 2^{<\kappa} = \kappa$, and the number of subsets of N of size smaller than κ is equal κ $\textcircled{e}$. We construct an elementary chain of size κ models $(M_\alpha)_{\alpha < \kappa}$ where M_0 is an arbitrary size κ model of T , and:

- M_δ , for a limit ordinal δ , is the union of M_β 's with $\beta < \delta$;
- $M_{\alpha+1}$ is a model that realises all types from X_{M_α} .

Let M be the union of all M_α 's. Any subset $A \subseteq M$ is contained in some M_α by the fact that $\text{cf}(\kappa) = \kappa$. Thus, any type over A will be realised in $M_{\alpha+1}$, hence in the constructed model M . $\square$

A cardinal κ satisfying the assumptions of the above theorem is called *strongly inaccessible*. Because of the Gödel's second incompleteness theorem, ZFC cannot prove existence of such κ . There are at least two solutions of this issue: (1) work in $\text{ZFC} \cup \{\text{"There exists a strongly inaccessible } \kappa"\}$, or (2) instead of saturated models work with κ -monster models, for a fixed κ . The reason why κ -monster models are convenient is the following, where by $\text{Aut}(M/A)$ we mean the set of automorphisms of M that fix A pointwise.

Proposition 7.8. Let $M \models T$ be a κ -monster model. Suppose that $A \subseteq M$ is a subset of size smaller than κ . Then:

- (1) For $\bar{a}, \bar{b} \in M^n$, $\text{tp}(\bar{a}/A) = \text{tp}(\bar{b}/A) \iff \exists \sigma \in \text{Aut}(M/A) \sigma(\bar{a}) = \bar{b}$.
- (2) Orbits of the action of $\text{Aut}(M/A)$ on M^n are in bijection with $S_n(A)$.

Proof. Exercise $\textcircled{e}$. $\square$

Fortunately, κ -monster models always exist.

Theorem 7.9. For every infinite cardinal κ , there exists a κ -monster model of T .

Proof. We will construct a κ -monster model M of T as a union of an elementary chain of models $(M_\alpha)_{\alpha < \kappa^+}$ with the following properties.

- If $\delta < \kappa^+$ is a limit ordinal, then M_δ is the union of the elementary chain $(M_\beta)_{\beta < \delta}$;
- $M_{\alpha+1}$ has the following properties with respect to M_α : (1) it is $|M_\alpha|^+$ -saturated, (2) every elementary bijection $f : A \rightarrow B$ between subsets $A, B \subseteq M_\alpha$ extends to an elementary bijection $f' : A' \rightarrow B'$ between subsets $A', B' \subseteq M_{\alpha+1}$ such that $M_\alpha \subseteq A', B'$.

For such a sequence we declare M to be the union of $(M_\alpha)_{\alpha < \kappa^+}$. Such M will satisfy the requirements of the theorem, because $\text{cf}(\kappa^+) = \kappa^+$ $\textcircled{e}$. Thus it is enough to ensure that we can find $M_{\alpha+1}$ in the inductive construction.

First note that from Proposition 7.3 (and the back-and-forth method) we can deduce $\textcircled{e}$ that (1) implies (2). Thus, we need to prove that M_α can be embedded into an $|M_\alpha|^+$ -saturated model. This follows from a construction similar to the one performed in the proof of Theorem 7.7, but without caring about the cardinality of the extension. $\square$

If T is a complete theory we usually implicitly fix a cardinal κ (bigger than all quantities that we are interested in) and then we fix a κ -monster model $\mathfrak{C} \models T$, that we just refer to as a monster model. A subset $A \subseteq \mathfrak{C}$ is called *small*, if $|A| < \kappa$. If one wants to make a monster model independent of a choice of a cardinal and essentially unique, one can work with class models, see [7].

Exercises, sheet 4. In the following exercises sometimes a theory is mentioned but not precisely defined (e.g. the theory of algebraically closed fields). It is then the task of the reader to axiomatize it in some language in the first-order way.

1. Let M be an infinite countable L -structure, where L is countable. Prove that $\text{Th}(M)$ is ω -categorical if and only if $\text{Aut}(M)$ has only finitely many orbits when acting on M^n for any natural n .

2. Show that a theory T eliminates quantifiers if and only if every formula of the form $\exists x \psi(x, \bar{y})$ is T -equivalent to a quantifier-free formula. Deduce the following criterion for quantifier elimination:

($\star$) For every L -structure A and embeddings $A \subseteq M$, $A \subseteq N$ into κ -saturated models (with respect to their respective complete theories) $M, N \models T$, for every $a \in M$, there is $b \in N$ such that $\text{qftp}(a/A) = \text{qftp}(b/A)$.

By $\text{qftp}(a/A)$ we mean the collection of quantifier-free formulas in $L(A)$ satisfied by a in M (similarly with $\text{qftp}(b/A)$).

3. Prove that the theory of vector spaces over a fixed field K eliminates quantifiers. Characterise its completions depending on K .
4. Prove that the theory ACF (algebraically closed fields) eliminates quantifiers. Prove that the characteristic p or 0 characterises its completions.
5. Show that if F is the algebraic closure of a finite field and $f : F^n \rightarrow F^n$ is a function that whose each coordinate is a polynomial, then injectivity of f implies surjectivity of f . Hint: look at intermediate finite fields.
6. Fix a degree d and a number of variables n . Note that the statement “every map $f : F^n \rightarrow F^n$ whose entries are polynomials of degree less or equal d that is injective is also surjective” can be encoded by a first-order sentence σ in the ring language, about the field F . Show that $\text{ACF}_0 \vdash \sigma$ if and only if $\text{ACF}_p \vdash \sigma$ is valid for big enough p . Conclude that $\mathbb{C} \models \sigma$, i.e., with the Ax-Grothendieck theorem.
7. An element $b \in B$ in a Boolean algebra B is called an *atom* if for every $b' \subset b$ (i.e., $\neg b' \vee b = 1$) we have $b' = 0$ or $b' = b$. Show that the theory of atomless Boolean algebras eliminates quantifiers, is complete, and ω -categorical.
8. Show that theories considered in the previous exercises are model companions of their universal theories (identify those).
9. Let M, N be homogeneous models of the same cardinality of a complete theory. Assume that M, N realise the same types from $S_n(\emptyset)$ for every natural n . Show that $M \cong N$. Hint: look at the proof of Proposition 7.3.
10. Prove Proposition 7.8.
11. Let $\varphi(\bar{x}, \bar{c})$ be a formula with parameters from a κ -monster model M and let $A \subseteq M$ be a subset of size smaller than κ . Assume that the set

$$\varphi(M, \bar{c}) := \{\bar{b} \in M^{|\bar{x}|} : M \models \varphi(\bar{b}, \bar{c})\}$$

is invariant under automorphisms of M over A (setwise). Show that there is a formula $\psi(\bar{x}, \bar{z})$ and parameters $\bar{a} \in A^{|\bar{z}|}$ such that $\varphi(M, \bar{c}) = \psi(M, \bar{a})$. Hint: use the restriction of types map $S_n(A\bar{c}) \rightarrow S_n(A)$.

8. MORLEYIZATION AND IMAGINARIES

Some systems of axioms use more than one type of primitive object. For example, Euclid's axioms are about points P , lines l , and the incidence relation between them $P \in l$. One can formalise this in the framework of multisorted (first-order) logic, where a language L consists of:

- Sort symbols $\{S_\alpha\}_{\alpha \in A}$.
- Function symbols $\{f_i\}_{i \in I}$, each equipped with its arity, which is a tuple $(\alpha_1, \dots, \alpha_n; \beta)$ for some $n \in \mathbb{N}$ and $\alpha_1, \dots, \alpha_n, \beta \in A$.
- Predicate symbols $\{P_j\}_{j \in J}$, each equipped with its arity which is a tuple $(\alpha_1, \dots, \alpha_m)$ for some $m \in \mathbb{N}$ and $\alpha_1, \dots, \alpha_m \in A$.

- Constant symbols $\{c_k\}_{k \in K}$, each equipped with a sort of index $\alpha = \alpha(k) \in A$.

In that case, an L -structure M is a collection of non-empty sets S_α^M together with interpretations of other symbols, where if f is a function symbol with arity $(\alpha_1, \dots, \alpha_n; \beta)$, then its interpretation is a function

$$f^M : S_{\alpha_1}^M \times \dots \times S_{\alpha_n}^M \rightarrow S_\beta^M.$$

Similarly, if P is a predicate symbol with arity $(\alpha_1, \dots, \alpha_m)$, then its interpretation is a subset

$$P^M \subseteq S_{\alpha_1}^M \times \dots \times S_{\alpha_m}^M.$$

If c is a constant of sort S , then M is equipped with its interpretation $c^M \in S^M$. When defining L -formulas, we need to consider variables decorated by sort symbols so each sort has a different set of variables associated to it.

Exercise 8.1. (e) Convince yourself that Henkin's proof of completeness works for multisorted structures.

The problem we will consider is how to present a theory “without coordinates”. Let us start with an example.

Example 8.2. Let L be a 2-sorted language with sorts P, Q , one binary relation $\in$ between P and Q , and three constant symbols c, d, e of the sort P . Consider an L -structure M where $P^M = \mathbb{R}^2$, $Q^M = \{\text{lines in } \mathbb{R}^2\}$, $c^M = (0, 0)$, $d^M = (0, 1)$, $e^M = (1, 0)$ (or any other three non-colinear points), and $\in^M$ is the relation of a point belonging to a line. Using Thales's theorem and other classical geometry facts, it is easy to define addition and multiplication on $\mathbb{R} \times \{0\} \subseteq M$, thus $Th(M)$ encodes all first-order statements true in $(\mathbb{R}, +, \cdot, 0, 1)$. On the other hand, working in the field language, it is convenient to define the set of lines in $\mathbb{R}^2$ as a quotient by a definable equivalence relation $\{(a, b, c) \in \mathbb{R}^3 : a \neq 0 \vee b \neq 0\} / E$ where

$$(a, b, c) E (a', b', c') \iff (\exists d)(d \neq 0 \wedge a' = da \wedge b' = db \wedge c' = dc)$$

and (a, b, c) represents the line $ax + by + c = 0$.

So what is the true form of real numbers? Is it the field, or the incidence geometry of lines in a plane? The answer I want to advertise is that we should not discriminate against any definable configuration, and it is good to have in mind all possible ones. To do that we will use category theory.

Definition 8.3. A *category* $\mathcal{C}$ consists of a collection (class) of objects $\text{Ob}_{\mathcal{C}}$ and for every $x, y \in \text{Ob}_{\mathcal{C}}$, a set of *morphisms* $\text{Mor}_{\mathcal{C}}(x, y)$, together with:

- identity morphisms $\text{id}_x \in \text{Mor}_{\mathcal{C}}(x, x)$ for every $x \in \text{Ob}_{\mathcal{C}}$,
- composition maps

$$\circ_{x,y,z} : \text{Mor}_{\mathcal{C}}(y, z) \times \text{Mor}_{\mathcal{C}}(x, y) \rightarrow \text{Mor}_{\mathcal{C}}(x, z),$$

for every $x, y, z \in \text{Ob}_{\mathcal{C}}$.

We drop the dependence on x, y, z in $\circ = \circ_{x,y,z}$. If $f \in \text{Mor}_{\mathcal{C}}(x, y)$, we sometimes write $f : x \rightarrow y$. These are required to satisfy the following axioms:

- $f \circ \text{id}_x = f$, $\text{id}_y \circ f = f$ for all $f : x \rightarrow y$.
- $(f \circ g) \circ h = f \circ (g \circ h)$ whenever $f : z \rightarrow w$, $g : y \rightarrow z$, $h : x \rightarrow y$.

In particular, if x, y are objects of a category $\mathcal{C}$, we call them *equivalent*, denoted $x \simeq y$, if there are morphisms $f : x \rightarrow y, g : y \rightarrow x$ such that $f \circ g = \text{id}_y, g \circ f = \text{id}_x$ (so that f has an inverse g , so it is an *isomorphism*). A morphism $f : x \rightarrow y$ is called an *epimorphism* (resp. *monomorphism*) if for every $g_1, g_2 : y \rightarrow z$ (resp. $g_1, g_2 : w \rightarrow x$) if $g_1 \circ f = g_2 \circ f$ (resp. $f \circ g_1 = f \circ g_2$) then $g_1 = g_2$.

Example 8.4. Verify $\textcircled{e}$ that the following are categories (by defining compositions and checking the conditions precisely).

- (1) Set is the common notation for the category of sets, where objects are sets, and morphisms between x and y are functions from x to y .
- (2) The collection of L -structures, for a first-order language L , where morphisms between M and N are homomorphisms.
- (3) The class of models of some first-order theory T , where as morphisms we take L -homomorphisms. Verify that groups or rings (with respective notions of homomorphisms) form a category of this form.
- (4) The class of models of some first-order theory T , where as morphisms we take elementary embeddings. We denote this category by $\text{Mod}(T)$.
- (5) If $(I, <)$ is a partial order, we can consider a category where $O = I$ and $M = \{(x, y) \in I^2 : x \leq y\}$, with domain and codomain maps being projections.

Definition 8.5. Let $\mathcal{C}, \mathcal{D}$ be categories. A *functor* between $\mathcal{C}$ and $\mathcal{D}$ is a (class) function $F : \text{Ob}_{\mathcal{C}} \rightarrow \text{Ob}_{\mathcal{D}}$ together with functions

$$F_{x,y} : \text{Mor}_{\mathcal{C}}(x, y) \rightarrow \text{Mor}_{\mathcal{D}}(F(x), F(y)),$$

for every pair of objects x, y of $\mathcal{C}$ (later on, we drop the dependence on x, y), such that $F(f \circ g) = F(f) \circ F(g)$ for all composable pairs of morphisms f, g in $\mathcal{C}$, and such that $F(\text{id}_x) = \text{id}_{F(x)}$ for all $x \in \text{Ob}_{\mathcal{C}}$. We usually write $F : \mathcal{C} \rightarrow \mathcal{D}$ to denote that F is a functor between $\mathcal{C}$ and $\mathcal{D}$.

Example 8.6. Verify $\textcircled{e}$ that the following are functors (define respective F 's on morphisms and verify the axioms precisely):

- (1) $F : \text{Ring} \rightarrow \text{Ab}$ sending a ring to the corresponding abelian group.
- (2) $F : \text{Set} \rightarrow \text{Grp}$ sending a set X to a group freely generated by X .

Definition 8.7. We define the category $\text{Def}(T)$ of *definable sets in T* , where objects are formulas $\varphi(\bar{x})$ and morphisms between $\varphi(\bar{x})$ and $\psi(\bar{y})$ are T -equivalence classes of formulas $[\xi(\bar{x}, \bar{y})]$ such that

$$T \vdash \xi(\bar{x}, \bar{y}) \rightarrow \varphi(\bar{x}) \wedge \psi(\bar{y})$$

and

$$T \vdash (\forall \bar{x})(\varphi(\bar{x}) \rightarrow (\exists! \bar{y})(\xi(\bar{x}, \bar{y}))),$$

where $\exists!$ is the “there exists unique” quantifier (define it precisely $\textcircled{e}$). If

$$[\xi(\bar{x}, \bar{y})] : \varphi(\bar{x}) \rightsquigarrow \psi(\bar{y}) \text{ and } [\zeta(\bar{y}, \bar{z})] : \psi(\bar{y}) \rightsquigarrow \eta(\bar{z})$$

are morphisms (we use $\rightsquigarrow$ not to confuse a morphism with an implication), we define their composition as the T -equivalence class of

$$\pi(\bar{x}, \bar{z}) = \exists \bar{y} \xi(\bar{x}, \bar{y}) \wedge \zeta(\bar{y}, \bar{z}).$$

The following definition is very similar, however quotients by equivalence relations are also allowed.

Definition 8.8. We define the category $\text{Int}(T)$ of *interpretable sets in T* , where objects are pairs $(\varphi(\bar{x}), E(\bar{x}, \bar{x}'))$ where $\varphi(\bar{x})$ is a formula and T proves that E is an equivalence relation on tuples satisfying φ (and does not hold otherwise, in particular φ is E -invariant). Morphisms between $\varphi/E = (\varphi(\bar{x}), E)$ and $\psi/F = (\psi(\bar{y}), F)$ are T -equivalence classes of formulas $[\xi(\bar{x}, \bar{y})]$ such that

$$T \vdash \xi(\bar{x}, \bar{y}) \rightarrow \varphi(\bar{x}) \wedge \psi(\bar{y}),$$

$$T \vdash E(\bar{x}, \bar{x}') \wedge F(\bar{y}, \bar{y}') \rightarrow (\xi(\bar{x}, \bar{y}) \leftrightarrow \xi(\bar{x}', \bar{y}')),$$

and

$$T \vdash (\forall \bar{x}) \left(\varphi(\bar{x}) \rightarrow \exists \bar{y} \left(\xi(\bar{x}, \bar{y}) \wedge (\forall \bar{y}') (\xi(\bar{x}, \bar{y}') \rightarrow F(\bar{y}, \bar{y}')) \right) \right).$$

If $[\xi(\bar{x}, \bar{y})] : (\varphi/E)(\bar{x}) \rightsquigarrow (\psi/F)(\bar{y})$ and $[\zeta(\bar{y}, \bar{z})] : (\psi/F)(\bar{y}) \rightsquigarrow (\eta/G)(\bar{z})$ are morphisms, we define their composition as the T -equivalence class of

$$\pi(\bar{x}, \bar{z}) = \exists \bar{y} \xi(\bar{x}, \bar{y}) \wedge \zeta(\bar{y}, \bar{z}).$$

We work with equivalence classes of formulas, because otherwise composition would not be associative. Note that every definable set can be treated as an interpretable one, when one considers the equality equivalence relation on it. Moreover, the set of morphisms between two definable sets is the same whenever we treat them as definable, or interpretable (with equalities as respective equivalence relations). Thus, we identify definable sets with a *full subcategory* of the category of interpretable sets. A *subcategory* of a category $\mathcal{C}$ consists of a subset of its objects $\mathcal{C}_0$ and a subset of morphisms between any two objects from $\mathcal{C}_0$, such that $\mathcal{C}_0$ is also a category when equipped with composition maps of $\mathcal{C}$, and $\mathcal{C}_0$ contains identity morphisms of its objects. We call $\mathcal{C}_0$ a *full subcategory*, if we take all possible morphisms from $\mathcal{C}$ between objects of $\mathcal{C}_0$.

Lemma 8.9. Let $M \models T$. Then M defines a functor $F^M : \text{Int}(T) \rightarrow \text{Set}$ where $(\varphi/E)(\bar{x})$ is taken to $F^M(\varphi) = (\varphi/E)(M) := \{[\bar{a}]_{E^M} : M \models \varphi(\bar{a})\}$ and $[\xi(\bar{x}, \bar{y})] : (\varphi/E)(\bar{x}) \rightsquigarrow (\psi/F)(\bar{y})$ is taken to $f : (\varphi/E)(M) \rightarrow (\psi/F)(M)$, where $f([\bar{a}]_{E^M}) = [\bar{b}]_{F^M}$ if and only if $M \models \xi(\bar{a}, \bar{b})$.

Proof. ⑥ Exercise. □

If $\varphi(\bar{x})$ is a formula we write $\varphi(M) \subseteq M^{|\bar{x}|}$ for $\{\bar{a} \in M^{|\bar{x}|} : M \models \varphi(\bar{a})\}$.

Lemma 8.10. $[\xi(\bar{x}, \bar{y})] : (\varphi/E)(\bar{x}) \rightsquigarrow (\psi/F)(\bar{y})$ is an epimorphism in $\text{Int}(T)$ if and only if

$$T \vdash (\forall \bar{y} : \psi(\bar{y})) (\exists \bar{x} : \varphi(\bar{x})) (\exists \bar{y}' : \psi(\bar{y}')) (\xi(\bar{x}, \bar{y}') \wedge F(\bar{y}, \bar{y}')).$$

Proof. Exercise ⑥. □

There are similar characterisation of monomorphisms and isomorphisms respectively ⑥. Note that we are using bounded quantifiers, but these are treated as shortcuts - one can easily express them in first-order logic ⑥.

Definition 8.11. Let $\mathcal{C}, \mathcal{D}$ be categories and assume that $F, G : \mathcal{C} \rightarrow \mathcal{D}$ are functors. A *natural transformation* between F and G , written $\eta : F \Rightarrow G$ consists of a $\mathcal{D}$ -morphism $\eta_c : F(c) \rightarrow G(c)$ such that for every $\mathcal{C}$ -morphism $f : c \rightarrow c'$ the following

diagram commutes

$$\begin{array}{ccc} F(c) & \xrightarrow{\eta_c} & G(c) \\ F(f) \downarrow & & \downarrow G(f) \\ F(c') & \xrightarrow{\eta_{c'}} & G(c') \end{array}$$

By commuting we mean that in $\mathcal{D}$ we have $G(f) \circ \eta_c = \eta_{c'} \circ F(f)$. A natural transformation η is *invertible* if all morphisms η_c are invertible in $\mathcal{D}$.

Note that if $M \prec M'$, then for every formula $\varphi(\bar{x})$ we have $\varphi(M) = \varphi(M') \cap M^{|\bar{x}|}$. Similarly, if $\alpha : M \rightarrow M'$ is an elementary map, then it maps $\varphi(M)$ to $\varphi(M')$.

Exercise 8.12. (e) Assume that $[\xi(\bar{x}, \bar{y})] : \varphi(\bar{x}) \rightsquigarrow \psi(\bar{y})$ is a morphism in $\text{Def}(T)$. Let $\alpha : M \rightarrow M'$ be an elementary map between $M, M' \models T$. Show that the following diagram commutes

$$\begin{array}{ccc} \varphi(M) & \xrightarrow{\alpha|_{\varphi}} & \varphi(M') \\ \downarrow & & \downarrow \\ \psi(M) & \xrightarrow{\alpha|_{\psi}} & \psi(M') \end{array}$$

where the vertical maps are induced by ξ in M and M' respectively. Verify a similar claim about interpretable sets.

In particular, an elementary map $\alpha : M \rightarrow M'$ defines a natural transformation $\eta_\alpha : F^M \Rightarrow F^{M'}$ between functors $F^M, F^{M'} : \text{Int}(T) \rightarrow \text{Set}$ (and between their restrictions to $\text{Def}(T)$).

Lemma 8.13. Let $M, M' \models T$ and let $\eta : F^M \Rightarrow F^{M'}$ be a natural transformation. Then there is an elementary map $\alpha : M \rightarrow M'$ such that $\eta = \eta_\alpha$.

Proof. Let $\alpha = \eta_{(x=x)}$ be the function $M \rightarrow M'$ that η induces on the definable set $(x = x)$. First let us check that α is elementary. Pick $a_1, \dots, a_n \in M$ and a formula $\varphi(x_1, \dots, x_n)$. Consider the following commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\eta_{(x=x)}} & M' \\ p_i \uparrow & & \uparrow p_i \\ M^n & \xrightarrow{\eta_{(\bar{x}=\bar{x})}} & M'^n \\ \uparrow & & \uparrow \\ \varphi(M) & \xrightarrow{\eta_\varphi} & \varphi(M') \end{array}$$

where vertical morphisms are images by F^M and $F^{M'}$ respectively, of the corresponding definable maps in $\text{Def}(T)$. Since the top horizontal map is α for any projection $i = 1, \dots, n$, it follows that $\eta_{(\bar{x}=\bar{x})} : M^n \rightarrow M'^n$ is applying α on each coordinate. The bottom vertical maps are induced by the morphism $[\varphi(\bar{x}) \wedge \bar{x} = \bar{y}] : \varphi(\bar{x}) \rightsquigarrow (\bar{y} = \bar{y})$ where $|\bar{y}| = n$. Thus $\eta_\varphi = \alpha^n|_{\varphi(M)}$. In particular we get

$$M \models \varphi(\bar{a}) \iff \bar{a} \in \varphi(M) \implies \alpha(\bar{a}) \in \varphi(M') \iff M' \models \varphi(\alpha(\bar{a})).$$

By considering $\neg\varphi(\bar{x})$ we obtain the reverse implication. Thus α is elementary. We leave it as an exercise that $\eta = \eta_\alpha$ (e). $\square$

Fix a language L and let T be an L -theory.

Definition 8.14. Let $L' = ((f_i)_{i \in I}, (P_j)_{j \in J}, (c_k)_{k \in K})$. An *interpretation* of (a one-sorted language) L' in a theory T consists of:

- 1) an L -formula $\varphi(x)$;
- 2) an L -formula $E(x, x')$, where x' has the same length as x ;
- 3) for every function symbol f of arity n in L' , an L -formula $\xi_f(x_1, \dots, x_n, x)$, where each variable has the same length as x ;
- 4) for every relation symbol P of arity m in L' , a formula $\chi_P(x_1, \dots, x_m)$, where each variable has the same length as x ;
- 5) for every constant symbol c in L' , a formula $\gamma_c(x)$.

For the ease of notation we write x instead of $\bar{x}$ for the variables of φ . The above data is required to satisfy:

- i. $T \vdash E(x, x') \rightarrow \varphi(x) \wedge \varphi(x')$, $T \vdash \exists x \varphi(x)$, and T proves that E is an equivalence relation on tuples satisfying φ .
- ii. $T \vdash \xi_f(x_1, \dots, x_n, x) \rightarrow \bigwedge_{i=1}^n \varphi(x_i) \wedge \varphi(x)$ and T proves that $\xi_f(x_1, \dots, x_n, x)$ defines a graph of a function of first n -variables modulo E (to the last variable modulo E), on $(n+1)$ -tuples $(x_1, \dots, x_n, x)$ such that all separately satisfy the formula φ .
- iii. $T \vdash \chi_P(x_1, \dots, x_m) \rightarrow \bigwedge_{i=1}^m \varphi(x_i)$ and T proves that χ_P depends only on the E -equivalence classes of the variables.
- iv. $T \vdash (\exists x)(\gamma_c(x)) \wedge (\forall x, x')(\gamma_c(x) \wedge \gamma_c(x') \rightarrow \varphi(x) \wedge \varphi(x') \wedge E(x, x'))$.

When $M \models T$, we equip $N := (\varphi/E)(M)$ with an L' -structure by defining f^N as having the graph $\xi_f(M)/E^M \subseteq (\varphi/E)(M)^{n+1}$, P^N as being $\chi_P(M)/E^M \subseteq (\varphi/E)(M)^m$, and c^N as being the unique class $[a]_{E^M} \in (\varphi/E)(M)$ such that $M \models \gamma_c(a)$. If T' is an L' -theory, we say that this data is an *interpretation of T' in T* if additionally, $N = (\varphi/E)(M) \models T'$ for every $M \models T$. When the definable relation E is just equality, we say that this interpretation exhibits T' as a *reduct of T* (or just that T' is a reduct of T).

If L' is a multisorted language, in the point 1) of the definition we fix an L -formula for every sort of L' .

Example 8.15. Let T be the theory of fields in $L = (+, \cdot, 0, 1)$ and let T' be the theory of groups in $L' = (\cdot, e)$. Let $\varphi(x_{11}, \dots, x_{1n}, x_{21}, \dots, x_{2n}, \dots, x_{nn})$ be the L -formula $\det([x_{ij}]_{i=1, \dots, n}^{j=1, \dots, n}) \neq 0$ and denote by $\hat{x}$ the tuple of variables x_{ij} . We define:

- $E(\hat{x}, \hat{x}')$ by $\hat{x} = \hat{x}' \wedge \varphi(\hat{x}) \wedge \varphi(\hat{x}')$,
- $\gamma_e(\hat{x})$ by $\bigwedge_{1 \leq i, j \leq n} x_{ij} = \delta_{ij}$, where δ_{ij} is the Kronecker delta,
- $\xi_{\cdot}(\hat{x}, \hat{y}, \hat{z})$ by the conjunction of $\varphi(\hat{x}) \wedge \varphi(\hat{y}) \wedge \varphi(\hat{z})$ and of the formula

$$\bigwedge_{1 \leq i, j \leq n} \left(z_{ij} = \sum_{k=1}^n x_{ik} \cdot y_{kj} \right).$$

In that case if $K \models T$, then $\varphi(K) \subseteq K^{n^2}$ defines the group $\mathrm{GL}_n(K)$. Thus the preceding data determines an interpretation of the theory of groups inside the theory of fields (in this case it is a reduct). If we instead take E to be the definable relation stating that $\hat{x}$ and $\hat{x}'$ lie in the same coset of the center of GL_n , then we get $\mathrm{PGL}_n(K)$ instead.

We defined interpretation of T' in T semantically, but note that it is possible to write it syntactically.

Exercise 8.16. © Fix the notation from Definition 8.14. Show that for every L' -formula $\sigma(z_1, \dots, z_n)$ there is an L -formula $\tilde{\sigma}(x_1, \dots, x_n)$ (each x_i is a tuple of variables of length $|x|$) such that for every model $M \models T$ and every $a_1, \dots, a_n \in \varphi(M)$ we have

$$N \models \sigma([a_1]_E, \dots, [a_n]_E) \iff M \models \tilde{\sigma}(a_1, \dots, a_n),$$

where $N = (\varphi/E)(M)$. Hint: do it for the atomic formulas and then induct based on the complexity of σ . Show that $\sigma \mapsto \tilde{\sigma}$ defines a functor $\tilde{\cdot} : \text{Int}(T') \rightarrow \text{Int}(T)$.

This exercise has the following consequence. Let $\alpha : M \rightarrow M'$ be an elementary embedding and let $\eta : F^M \Rightarrow F^{M'}$ be the corresponding natural transformation. Note that $F^M, F^{M'} : \text{Def}(T) \rightarrow \text{Set}$ and by precomposing them with $\tilde{\cdot}$ we get functors $G^{\varphi(M)}, G^{\varphi(M')} : \text{Def}(T') \rightarrow \text{Set}$ and a natural transformation $\mu : G^{\varphi(M)} \Rightarrow G^{\varphi(M')}$. By Lemma 8.13 for the theory T' , μ is induced by the L' -elementary map $\alpha|_{\varphi} : \varphi(M) \rightarrow \varphi(M')$. The following is a more hands-on proof.

Lemma 8.17. Fix an interpretation of T' in T where the main sort of T' corresponds to a pair $(\varphi(x), E(x, x'))$ in the language of T . Then $F : \text{Mod}(T) \rightarrow \text{Mod}(T')$ defined by $F(M) = (\varphi/E)(M)$ is a functor.

Proof. Note that for an elementary $\alpha : M \rightarrow M'$ it needs to map $(\varphi/E)(M)$ to $(\varphi/E)(M')$. We need to check that $\alpha|_{\varphi} : (\varphi/E)(M) \rightarrow (\varphi/E)(M')$ is elementary. Let $\sigma(z_1, \dots, z_n)$ be an L' -formula and let $[a_1]_E, \dots, [a_n]_E \in (\varphi/E)(M)$. Let $\tilde{\sigma}(x_1, \dots, x_n)$ be the formula from Exercise 8.16. We get that

$$\begin{aligned} (\varphi/E)(M) \models \sigma([a_1]_E, \dots, [a_n]_E) &\iff M \models \tilde{\sigma}(a_1, \dots, a_n) \\ \iff M' \models \tilde{\sigma}(\alpha(a_1), \dots, \alpha(a_n)) &\iff (\varphi/E)(M') \models \sigma(\alpha([a_1]_E), \dots, \alpha([a_n]_E)). \end{aligned}$$

□

Definition 8.18. Let $\mathcal{C}, \mathcal{D}$ be categories. We say that they are *equivalent* if there are functors $F : \mathcal{C} \rightarrow \mathcal{D}$ and $G : \mathcal{D} \rightarrow \mathcal{C}$ and invertible natural transformations $\alpha : \text{id}_{\mathcal{D}} \Rightarrow FG$, $\beta : \text{id}_{\mathcal{C}} \Rightarrow GF$.

Exercise 8.19. © A functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is called:

- *full* if for any $x, x' \in \text{Ob}_{\mathcal{C}}$ the function $F : \text{Mor}_{\mathcal{C}}(x, x') \rightarrow \text{Mor}_{\mathcal{D}}(F(x), F(x'))$ is surjective.
- *faithful* if for any $x, x' \in \text{Ob}_{\mathcal{C}}$ the function $F : \text{Mor}_{\mathcal{C}}(x, x') \rightarrow \text{Mor}_{\mathcal{D}}(F(x), F(x'))$ is injective.
- *essentially surjective* if for every $y \in \text{Ob}_{\mathcal{D}}$ there is $x \in \text{Ob}_{\mathcal{C}}$ such that $y \simeq F(x)$.

Show that there is a functor $G : \mathcal{D} \rightarrow \mathcal{C}$ such that F and G witness an equivalence between $\mathcal{C}$ and $\mathcal{D}$ if and only if F is *fully faithful* (which means both full, and faithful) and essentially surjective.

Definition 8.20. Consider an L -theory T and an L' -theory T' . We call those *bi-interpretable* if there are interpretations of T in T' and of T' in T such that the induced functors $\text{Int}(T) \rightleftarrows \text{Int}(T')$ exhibit an equivalence of categories.

If we use more category theory, we could phrase this definition in the language of Boolean pretopoi. For more details about this point of view see [4].

Definition 8.21. A *Morleyization* of T is the following theory T_{Mor} in the language $L_{\text{Mor}} = L \cup (R_\varphi(x) : \varphi(x) \in \text{Ob}_{\text{Def}(T)})$, where $R_\varphi(x)$ is of arity $|x|$:

- (1) $R_\chi \leftrightarrow \chi$ for atomic formulas χ ,
- (2) $R_{\neg\varphi} \leftrightarrow \neg R_\varphi$,
- (3) $R_{\varphi \wedge \psi} \leftrightarrow R_\varphi \wedge R_\psi$,
- (4) $R_{\exists x \psi(x,y)}(y) \leftrightarrow \exists x R_\psi(x,y)$,
- (5) R_σ for every $\sigma \in T$.

The *expansion by imaginaries* of T is the following theory T^{eq} in the multisorted language L^{eq} with sorts $(S_{\varphi/E} : \varphi/E \in \text{Ob}_{\text{Int}(T)}^*)$, where by $\text{Ob}_{\text{Int}(T)}^*$ we mean the collection of consistent interpretable sets (the ones where $T \vdash \exists x \varphi(x)$), and with:

- a function symbol $f_{[\xi(x,y)]} : S_{\varphi(x)/E} \rightarrow S_{\psi(y)/F}$ whenever

$$[\xi(x,y)] : \varphi(x)/E \rightsquigarrow \psi(y)/F$$

is a morphism in $\text{Int}(T)$.

- a predicate symbol $R_{\varphi'(x)/E}$ on the sort $S_{\varphi(x)/E}$, whenever T proves that φ' is E -invariant and $T \vdash \varphi'(x) \rightarrow \varphi(x)$.

The axioms of T^{eq} are:

- (1) $f_{[\xi(x,y)]}(a) = b \leftrightarrow R_{\xi(x,y)/x=x \wedge y=y}(a,b)$ for every $[\xi(x,y)] : \varphi(x)/E \rightsquigarrow \psi(y)/F$ among morphisms of $\text{Int}(T)$, where $R_{\xi(x,y)/x=x \wedge y=y}$ is a predicate on the sort $S_{x=x \wedge y=y}$.
- (2) $(f_{[\varphi(y) \wedge \varphi(x) \wedge x=y]}(c) = a) \rightarrow (R_{\varphi'(x)/E}(a) \leftrightarrow R_{\varphi'(x)}(c))$ for every morphism $[\varphi(y) \wedge \varphi(x) \wedge x=y] : \varphi(x) \rightsquigarrow \varphi(y)/E$ where φ' is E -equivariant $T \vdash \varphi'(x) \rightarrow \varphi(x)$.
- (3) axioms from T_{Mor} interpreted in the sorts where the equivalence relation is the identity.

In particular, we treat $L \subseteq L_{\text{Mor}} \subseteq L^{\text{eq}}$ and $T \subseteq T_{\text{Mor}} \subseteq T^{\text{eq}}$.

Exercise 8.22. (e) Prove that T^{eq} is interpretable in T . Show that both T_{Mor} and T^{eq} eliminate quantifiers.

In the following proof, for simplicity we work with saturated models. One can also give a proof only using ZFC.

Theorem 8.23 (Makkai's conceptual completeness). An interpretation $j = (\varphi(x), E, \dots)$ of T' in T induces an equivalence of categories $F_j : \text{Mod}(T) \rightarrow \text{Mod}(T')$ if and only if j witnesses that T and T' are bi-interpretable.

Proof. Let $j_* : \text{Int}(T') \rightarrow \text{Int}(T)$ be the functor induced by j , constructed in Exercise 8.16. It is enough to show that it is an equivalence of categories. In the proof we use single letters for tuples of variables. We show fully faithfulness and essential surjectivity of j_* in (A) and (B) respectively.

(A) Showing that j_* is fully faithful: Assume that $\chi(z)$ is an L' -formula and let $\psi(x) = (j_*\chi)(x)$. Let $S_\chi(T') \subseteq S_z(T')$ be the clopen subspace consisting of T' -types containing the formula $\chi(z)$ and let $S_\psi(T) \subseteq S_x(T)$ be the clopen subspace consisting of T -types containing the formula $\psi(x)$. The functor j_* induces a continuous function $\pi : S_\psi(T) \rightarrow S_\chi(T')$. We first show that this function is a homeomorphism. Surjectivity follows from the fact that F_j is essentially surjective: if $p \in S_\chi(T')$ is realised in a model $M' \models T'$ by $a \in M'^{|z|}$, then if $M \models T$ is such that $F_j(M) \cong M'$ and $b \in M^{|x|}$ is a tuple corresponding to a , then $\pi(\text{tp}(b)) = \text{tp}(a) = p$.

For injectivity assume that $p \in S_\chi(T')$ can be written as $\pi(p_1) = \pi(p_2) = p$ for some $p_1, p_2 \in S_\psi(T)$. We have two cases: (1) p_1 and p_2 imply two different completions of T - then we get a contradiction with F_j being an equivalence of categories, as connected components of $\text{Mod}(T)$ and $\text{Mod}(T')$ correspond to completions of T and T' respectively $\textcircled{c}$; (2) p_1 and p_2 imply the same completion of T - in the case we can find $b_1 \models p_1, b_2 \models p_2$ in a saturated model $M \models T$. Then $F_j(M)$ is a saturated model of T' and thus there is an automorphism f of $F_j(M)$ mapping b_1 to b_2 . Since

$$F_j : \text{Aut}(M) \rightarrow \text{Aut}(F_j(M))$$

is an isomorphism, by the fully faithfulness of F_j , we get that f lifts to an automorphism of M , and hence $p_1 = p_2$. Thus we conclude that $\pi : S_\psi(T) \rightarrow S_\chi(T')$ is a continuous bijection between compact Hausdorff spaces, therefore a homeomorphism. We leave it as an exercise to show that π being a homeomorphism (for every pair χ, ψ as above) implies that j_* is fully faithful $\textcircled{c}$.

(B) Showing that j_* is essentially surjective: Consider a T -definable set $\psi(y)$. We claim that there is a T' -definable set $\xi(z)$ and an epimorphism $j_*\xi \rightsquigarrow \psi$ in $\text{Int}(T)$. This is enough to prove (B) by looking at the fiber equivalence relation on $j_*\xi$ and using (A). If the desired epimorphism does not exist, consider the following collection of formulas

$$p(y) = \{\psi(y)\} \cup \{\neg\theta(y) : \theta \in I\},$$

where $\theta \in I$ if θ is an L -formula in variables y such that for some morphism $[\eta(z, y)] : (j_*\xi)(z) \rightsquigarrow \psi(y)$ in $\text{Int}(T)$, $\theta(y) = (\exists z)((j_*\xi)(z) \wedge \eta(z, y))$. By our assumption p is a (partial) type. Let $M \models T$ be such that $c \in M^{|y|}$ realises p . Consider the multisorted language $L_\pm = L^{\text{eq}} \cup \{a_+ : a \in M^{\text{eq}}\} \cup \{a_- : a \in M^{\text{eq}}\}$ and the $L_\pm$ -theory

$$T_\pm = \text{Diag}(M^{\text{eq}})_+ \cup \text{Diag}(M^{\text{eq}})_- \cup \{c_+ \neq c_-\} \cup \{a_+ = a_- : "a \in F_j(M)"\},$$

where by we include two copies of $\text{Diag}(M^{\text{eq}})$, one with constants decorated by $+$ and the other one with constants decorated by $-$, and where by " $a \in F_j(M)$ " we mean elements corresponding to the sorts declared in j as the universe of an L' -structure. Note that if $T_\pm$ was consistent, then for a model $M_\pm \models T_\pm$ we would get two elementary embeddings $h_+ : M \rightarrow M_\pm, h_- : M \rightarrow M_\pm$ that map m to m_+ and m_- respectively, and they would be different as $h_+(c) = c_+ \neq c_- = h_-(c)$. On the other hand, $F_j(h_+) = F_j(h_-)$ which gives a contradiction. Hence we get that $T_\pm$ is inconsistent, and by compactness it means that there is an L^{eq} -formula $\mu(x, y, z)$ such that

$$T^{\text{eq}} \cup \{\mu(a, b_+, c_+)\} \cup \{\mu(a, b_-, c_-)\} \cup \{c_+ \neq c_-\}$$

is inconsistent, where by a we mean one of $a_+ = a_-$. In particular, this means that T^{eq} proves that $\exists z \mu(r, z, w)$ constitutes a definable function from the sort $r \in S$ declared by j as the universe of an L' -structure, to the L -sort corresponding to variables y of the L -formula $\psi(y)$. Moreover, $M^{\text{eq}} \models \mu(a, b, c)$, so in M^{eq} the definable function $\exists z \mu(r, z, w)$ sends a to c . This contradicts the defining property of $c \in M^{|y|}$. Thus, we are done with proving (B). $\square$

As a consequence we get the following ‘universal’ property of T^{eq} .

Theorem 8.24. For every L' -theory T' and an interpretation j of T in T' that induces an equivalence $F_j : \text{Mod}(T') \rightarrow \text{Mod}(T)$, it is possible to exhibit T' as a reduct of T^{eq} .

Because of the possibility of working with new sorts for products, we usually drop the overline when denoting a tuple.

Exercises, sheet 5.

1. Solve Exercise 8.1.
2. Prove Lemma 8.10.
3. Solve Exercise 8.16.
4. Solve Exercise 8.22. In particular, if $M \models T$, then we denote by $M^{\text{eq}} \models T^{\text{eq}}$ its unique (up to a unique isomorphism) lift to a model of T^{eq} .
5. Show that if $M \models T$ is κ -saturated and $N = (\varphi/E)(M)$ is an L' -structure interpreted in M , then N is also κ -saturated.
6. Let T' be an L' -theory, with $L' = L \cup \{P\}$ for some n -ary predicate P . Assume that if $A, B \models T'$ are such that $A|_L = B|_L$, then for all $c \in A^n$ we have

$$A \models P(c) \iff B \models P(c).$$

Show that there is an L -formula $\varphi(x_1, \dots, x_n)$ such that

$$T' \vdash P(x_1, \dots, x_n) \leftrightarrow \varphi(x_1, \dots, x_n).$$

This result is called Beth definability theorem.

7. Let $\mathfrak{C}$ be a κ -monster model of a complete theory T and let $\varphi(x, a)$ be an L -formula. Show that there is an element $a' \in \mathfrak{C}^{\text{eq}}$ such that for all $\sigma \in \text{Aut}(\mathfrak{C})$ we have

$$\sigma(\varphi(\mathfrak{C}, a)) = \varphi(\mathfrak{C}, a) \iff \sigma(a') = a'.$$

Show that there is an L^{eq} -formula $\psi(x, y)$ such that $\varphi(\mathfrak{C}, a) = \psi(\mathfrak{C}, a')$. If we write $X = \varphi(\mathfrak{C}, a) \subseteq \mathfrak{C}^x$, an element a' with the considered property is called a *canonical parameter* of X .

8. Show that if a', a'' are canonical parameters of a definable set $X \subseteq \mathfrak{C}^x$, then there is a definable function f whose domain contains a' such that $f(a') = a''$.

9. EHRENFUCHT–MOSTOWSKI MODELS

Because of the Löwenheim–Skolem theorem it is impossible to find a theory with a model in only one infinite cardinality. It may feel paradoxical that theories like the Peano Arithmetic or ZFC have many models. Some of these models are rigid, in the sense that they have no non-trivial automorphisms, e.g. $\mathbb{N}$ as a model of PA. But is every model of PA rigid? Is there a theory T with infinite models that are all rigid? Ehrenfeucht and Mostowski showed that the answer to these questions is ‘no’ and furthermore that the automorphisms group of a model can be required to be very big. In course of constructing such models we will introduce Skolemization and indiscernibles, which will be very useful later.

Definition 9.1. We say that T has *built-in Skolem functions* if for every L -formula $\varphi(\bar{x}, y)$ there is a function symbol f of arity $\bar{x}$ such that

$$T \vdash (\forall \bar{x}) \left((\exists y \varphi(\bar{x}, y)) \rightarrow \varphi(\bar{x}, f(\bar{x})) \right).$$

The Skolem functions here are ‘built-in’ in the sense that we have function symbols for them. We skip the ‘build-in’ in the definition, if f ’s are just required to be definable functions with respect to the theory T . For example, Peano Arithmetic has Skolem functions that are not built-in, that can be defined by considering the smallest element satisfying a formula (although models of PA are not necessarily well-ordered).

Definition 9.2. The *Skolemization* of T is the following theory T_{Sk} in the language L_{Sk} . First, for any theory T_0 in a language L_0 define

$$L'_0 = L_0 \cup \{t_{\varphi(\bar{x}, y)}(\bar{x}) : L_0 - \text{formulas } \varphi(\bar{x}, y)\},$$

where $t_{\varphi(\bar{x}, y)}(\bar{x})$ is a new function symbol of arity $\bar{x}$. We also define T'_0 as T together with the axioms

$$(\forall \bar{x}) \left((\exists y \varphi(\bar{x}, y)) \rightarrow \varphi(\bar{x}, t_{\varphi(\bar{x}, y)}(\bar{x})) \right),$$

for every L_0 -formula $\varphi(\bar{x}, y)$. We define L_{Sk} as the increasing union of $L^{(n)}$, $n \in \mathbb{N}$, and T_{Sk} as the increasing union of $T^{(n)}$ for $n \in \mathbb{N}$.

Note that if $M \models T$, then there is an extension of M to an L_{Sk} -structure M_{Sk} so that $M_{\text{Sk}} \models T_{\text{Sk}}$. Indeed, we choose a well-ordering on M and interpret the new function symbols as taking the smallest witness for validity of the ambient formula.

Definition 9.3. Assume that we are given an extension $L \subseteq L'$ and $T \subseteq T'$. We call it *conservative* if for any L -sentence σ we have

$$T \vdash \sigma \iff T' \vdash \sigma.$$

The proof of the following theorem shows that conservativity is equivalent to the fact that every model $M \models T$ lifts to a model $M' \models T'$.

Theorem 9.4. T_{Sk} has build-in Skolem functions. Moreover, $T \subseteq T_{\text{Sk}}$ is a conservative extension.

Proof. We only prove the latter statement. The implication from left to right is automatic. On the other hand, if $T \not\models \sigma$, then there is an $M \models T \cup \{\neg\sigma\}$ and we can lift M to a model of $M_{\text{Sk}} \models T_{\text{Sk}} \cup \{\neg\sigma\}$, so $T_{\text{Sk}} \not\models \sigma$. $\square$

Now we introduce indiscernibles. In philosophy, these were introduced by Leibniz, as a principle that two distinct things cannot have all their properties in common.

Definition 9.5. A sequence $(a_i)_{i \in \mathbb{N}}$ in a model M of T is called an *indiscernible sequence* if for every $i_1 < \dots < i_n$ and $j_1 < \dots < j_n$ we have

$$(a_{i_1}, \dots, a_{i_n}) \equiv (a_{j_1}, \dots, a_{j_n}),$$

where $a \equiv b$ means $\text{tp}(a) = \text{tp}(b) \in S_n(T)$. We say that $(a_i)_{i \in \mathbb{N}}$ is an *indiscernible set* if for every pairwise distinct $i_1, \dots, i_n \in \mathbb{N}$ and $j_1, \dots, j_n \in \mathbb{N}$ we have

$$(a_{i_1}, \dots, a_{i_n}) \equiv (a_{j_1}, \dots, a_{j_n}).$$

If $A \subseteq M$ is a subset, then we say that $(a_i)_{i \in \mathbb{N}}$ is an *indiscernible sequence over A* if for every $i_1 < \dots < i_n$ and $j_1 < \dots < j_n$ we have

$$(a_{i_1}, \dots, a_{i_n}) \equiv_A (a_{j_1}, \dots, a_{j_n}),$$

where $a \equiv_A b$ means $\text{tp}(a/A) = \text{tp}(b/A) \in S_n(A)$ (similarly for indiscernible sets).

The definition is the same if we replace $(\mathbb{N}, <)$ with another linear order.

To produce indiscernibles we need Ramsey's theorem. Let $k, l \in \mathbb{N}$.

Theorem 9.6 (Ramsey's theorem). Let (V, E) be a graph with an infinite set of vertices. Then there is an infinite subset $V_0 \subseteq V$ such that V_0 is a clique or an anti-clique.

More generally, if V is an infinite set whose k -element subsets are coloured with l -colours, then there is an infinite $V_0 \subseteq V$ such that all k -element subsets of V_0 are coloured in the same way (we call V_0 monochromatic).

Proof. We prove it inductively based on the number of colours. If we only use one colour the conclusion is obvious. Now we pass to the induction step.

Pick a vertex p_1 . Consider a new colouring of $(k-1)$ -element subsets of $V \setminus \{p_1\}$ where a subset A has the colour of Ap_1 . By the inductive hypothesis, there is an infinite subset $V_1 \subseteq V \setminus \{p_1\}$ such that all $(k-1)$ -element subsets A of V_1 have the same colour, say i_1 . Take a vertex $p_2 \in V_1$. Again, there is an infinite subset $V_2 \subseteq V_1 \setminus \{p_2\}$ such that for all $(k-1)$ -element subsets B of V_2 , Bp_2 has colour i_2 . We proceed by constructing a sequence (p_k, V_k, i_k) where for every $(k-1)$ -element subset C of V_k , Cp_k has colour i_k , and $V_{k+1} \subseteq V_k \ni p_{k+1}$. Choose a subsequence $p_{j_1}, p_{j_2}, \dots$ such that their associated colours are the same. Then the set $\{p_{j_a} : a \in \mathbb{N}\} \subseteq V$ is monochromatic. $\square$

Definition 9.7. Let $A \subseteq M$ be a subset of a model M . The *definable closure* of A is the subset $\text{dcl}(A) \subseteq M$ consisting of elements $b \in M$ such that there is an L -formula $\varphi(x, y)$ and $a \in A^y$ for which

$$M \models \exists! x \varphi(x, b) \wedge \varphi(a, b).$$

The (model theoretic) *algebraic closure* of A , denoted $\text{acl}(A) \subseteq M$ consists of elements $b \in M$ that satisfy an *algebraic* $L(A)$ -formula $\varphi(x, a)$, i.e., a formula such that for some natural n , we have

$$M \models \exists^{\leq n} x \varphi(x, a).$$

A type $p \in S_x(A)$ is called *algebraic* if it contains an algebraic formula.

Note that if $A \subseteq M \prec N$, then the definable and algebraic closures of A in M and in N coincide $\textcircled{e}$. That is why in the notation the model M does not appear (we write $\text{dcl}(A)$ and not $\text{dcl}_M(A)$).

Exercise 9.8. $\textcircled{e}$ Show that an algebraic type $p \in S_x(A)$ is isolated. Show that the other implication does not hold in general.

Exercise 9.9. $\textcircled{e}$ Show that for $A \subseteq \mathfrak{C}$ we have:

$$(1) \ b \in \text{dcl}(A) \iff \text{Aut}(\mathfrak{C}/A) \cdot b = \{b\}, \text{ if and only if}$$

$$(\exists [\xi] : \varphi(x) \rightsquigarrow \psi(y))(\exists a \in A^x)([\xi](a) = b),$$

$$(2) \ b \in \text{acl}(A) \iff \text{Aut}(\mathfrak{C}/A) \cdot b \text{ is finite.}$$

Note that the same holds for $A \subseteq \mathfrak{C}^{\text{eq}}$. We denote by $\text{dcl}^{\text{eq}}(A)$ (resp. $\text{acl}^{\text{eq}}(A)$) the definable (resp. algebraic) closure calculated in $\mathfrak{C}^{\text{eq}}$.

Theorem 9.10. Let $\mathfrak{C}$ be a monster model of a complete theory and let $p \in S_1(A)$ be a non-algebraic type over a small set $A \subseteq \mathfrak{C}$. There is a non-constant indiscernible sequence $(a_i)_{i \in \mathbb{N}}$ in $\mathfrak{C}$ such that $\text{tp}(a_1/A) = p$.

Proof. Consider the following $(L(A) \cup \{x_i : i \in \mathbb{N}\})$ -theory

$\text{Diag}_{\text{el}}(A) \cup p(x_1) \cup \{\varphi(x_{i_1}, \dots, x_{i_n}) \leftrightarrow \varphi(x_{j_1}, \dots, x_{j_n}) : \bar{i}, \bar{j} \text{ increasing sequences in } \mathbb{N}\},$

where φ varies among all $L(A)$ -formulas and n varies among all natural numbers. It is enough that this theory is consistent, as $\mathfrak{C}$ is a monster model of T . Consider finitely many formulas $\varphi_1, \dots, \varphi_k$ each with n -variables $x_1, \dots, x_n$. Fix a linear order on $p(\mathfrak{C})$ and colour n -element subsets of $p(\mathfrak{C})$ into 2^k colours, where the colour of $\{a_1 < \dots < a_n\}$ is $(i_1, \dots, i_k) \in 2^k$ when for each $j = 1, \dots, k$ we have

$$\mathfrak{C} \models \varphi_j(a_1, \dots, a_n) \iff i_j = 0.$$

As p is non-algebraic, $p(\mathfrak{C})$ is infinite. By Ramsey's theorem there is an infinite subset $V_0 \subseteq p(\mathfrak{C})$ that is monochromatic with respect to this colouring. This shows that the part of the consider theory where we fix $n \in \mathbb{N}$ is consistent. By compactness we get that the whole theory is consistent, which finishes the proof. $\square$

Note that literally the same proof works if we replace $(\mathbb{N}, <)$ by any other infinite linear order, e.g. by $(\mathbb{Q}, <)$.

Theorem 9.11 (Ehrenfeucht-Mostowski). Let T be a complete theory in a countable language L , with infinite models. There is a countable model M of T such that $\text{Aut}(\mathbb{Q}, <)$ embeds into $\text{Aut}(M)$.

Proof. First, replace T, L by $T_{\text{Sk}}, L_{\text{Sk}}$. Note that it is enough to prove the theorem for some completion of T_{Sk} , as every automorphism of $N \models T_{\text{Sk}}$ is also an automorphism of its L -reduct. Thus we can without loss of generality assume that T has (build-in) Skolem functions. In particular, if $A \subseteq M \models T$, then $\text{dcl}(A)$ satisfies the Tarski-Vaught test, so $\text{dcl}(A) \prec M$.

Let $p \in S_1(\emptyset)$ be a non-algebraic type with respect to T . Such exists, as otherwise the theory would not have infinite models. Let $M \models T$ be a model where there is an indiscernible sequence $(a_i)_{i \in \mathbb{Q}}$. Define $H := \text{dcl}(\{a_i : i \in \mathbb{Q}\}) \prec M$. Note that every automorphism of $(\mathbb{Q}, <)$ induces an elementary self-bijection on $\{a_i : i \in \mathbb{Q}\}$ which extends uniquely to an automorphism of $H \text{ @ } \mathfrak{C}$. This is the desired embedding of $\text{Aut}(\mathbb{Q}, <)$ into $\text{Aut}(H)$. $\square$

Exercises, sheet 6.

1. Show that if T has Skolem functions then it eliminates quantifiers.
2. Use compactness and the infinitary Ramsey's theorem to prove the finitary version: for every finite n, k, l there is a natural number N such that for all sets V of cardinality $> N$ whose k -element subsets are coloured in l colours, there is a subset $V_0 \subseteq V$ of cardinality $\geq n$ which is monochromatic.
3. Solve Exercise 9.8.
4. Solve Exercise 9.9.
5. Convince yourself that Theorem 9.10 works for any linear order in place of $(\mathbb{N}, <)$.
6. For cardinals $\kappa, \mu, \nu, \lambda$ we write $\kappa \rightarrow (\mu)_\lambda^\nu$ if for every colouring of ν -sized subsets of κ using λ -many colours, there is a monochromatic subset of κ of size at least μ . Prove that $(2^\kappa)^+ \rightarrow (\kappa^+)_\kappa^2$.
7. Prove the Erdős-Rado theorem: for an infinite cardinal κ and a natural number $r \geq 0$ we have $\exp_r(\kappa)^+ \rightarrow (\kappa^+)_\kappa^{r+1}$, where $\exp_r(\lambda) = 2^{\exp_{r-1}(\lambda)}$ and $\exp_0(\lambda) = \lambda$.

10. PRIME MODELS

Let T be a complete theory in a language L .

Definition 10.1. A model $M \models T$ is called:

- (1) *prime* if for every $N \models T$ there is an elementary map $f : M \rightarrow N$,
- (2) *atomic* if for every finite $a \subseteq M$, $\text{tp}(a)$ is isolated.

For example, the cardinality $\aleph_0$ model of an ω -categorical theory is prime and atomic. In general, prime models may not exist - this happens e.g. for T is the theory of $\mathbb{Q}$ in the language with predicates $P_{a,b}$ interpreted as (a, b) , for every $a < b \in \mathbb{Q}$ (we will soon see why). We will show that the so-called ω -stable theories always have prime models.

Exercise 10.2. (e) Show that for a countable language L , if a prime model of T exists, it is atomic.

Fix a monster model $\mathfrak{C} \models T$.

Lemma 10.3. Let a, b be finite tuples in $M \models T$. Then $\text{tp}(a, b)$ is isolated if and only if $\text{tp}(a/b)$ is isolated and $\text{tp}(b)$ is isolated.

Proof. Embed M in $\mathfrak{C}$ and work with orbits of the automorphism group there. Let $\psi(x, y)$ be a formula isolating $\text{tp}(a, b)$. We claim that $\exists x \psi(x, y)$ isolates $\text{tp}(b)$, and $\psi(x, b)$ isolates $\text{tp}(a/b)$. Indeed, if $b' \models \exists x \psi(x, y)$, then there is a' such that $(a', b') \models \psi$, and by the assumption there is an automorphism that takes (a', b') to (a, b) , hence $\text{tp}(b') = \text{tp}(b)$. If $a'' \models \psi(x, b)$, then by the assumption there is an automorphism that takes (a'', b) to (a, b) , so $\text{tp}(a''/b) = \text{tp}(a/b)$. On the other hand, if $\alpha(x, b)$ isolates $\text{tp}(a/b)$ and $\beta(y)$ isolates $\text{tp}(b)$, then $\alpha(x, y) \wedge \beta(y)$ isolates $\text{tp}(a, b)$, because if $(a', b') \models \alpha(x, y) \wedge \beta(y)$, then we can find automorphisms that map b' to b (and say a' to a''), and that map (a'', b) to (a, b) . $\square$

Lemma 10.4. If $M \models T$ is countable and atomic, then M is prime.

Proof. Let $N \models T$. We will construct an elementary embedding $M \rightarrow N$. Enumerate $M = \{a_n : n \in \mathbb{N}\}$ and assume we have defined a partial elementary map $f : \{a_1, \dots, a_i\} \rightarrow N$ (i can be zero). Let $p = \text{tp}(a_{i+1}/a_1, \dots, a_i)$ and let $q = f(p)$. Type p is isolated by Lemma 10.3, so q is also isolated, and hence there is $b \models q$ in N . Extension $f \subseteq f \cup \{(a_{i+1}, b)\}$ is elementary and by taking the union of the chain of such extension we get the desired elementary embedding. $\square$

Note that a similar back-and-forth argument proves the following.

Corollary 10.5. If a countable atomic model of T exists, it is unique up to an isomorphism. Moreover, in that case it is prime and homogeneous.

Note that $(\mathbb{R}, <)$ is atomic, but not prime, so the countability assumption is necessary.

Definition 10.6. In the following κ is an infinite cardinal. Assume that L is countable. We call T :

- (1) κ -stable if for all subsets $A \subseteq M \models T$ of cardinality at most κ , we have $|S_1(A)| \leq \kappa$.
- (2) stable, if it is κ -stable for some κ .
- (3) superstable, if it is λ -stable, for all $\lambda \geq \kappa$, for some κ .

(4) *totally transcendental*, if it is $\aleph_0$ -stable (sometimes it is also called ω -stable).

Really, countability of the language only plays a role in the definition of ω -stability. Later we will give a different definition of being totally transcendental for any language, that is equivalent to this one in the case of countable L .

Theorem 10.7. An ω -stable theory is κ -stable for all infinite cardinals κ .

Proof. Assume that T is not κ -stable. Choose $A \subseteq M \models T$ with $|S_1(A)| > \kappa$. We will build a tree of formulas that will witness non ω -stability.

Call a formula $\varphi(x) \in L(A)$ big, if $||[\varphi]|| > \kappa$. First, we prove that if φ is big, then there is $\psi \in L(A)$ such that $\varphi \wedge \psi$ and $\varphi \wedge \neg\psi$ are both big. Assume that it was not the case. Then we can consider

$$p(x) = \{\psi(x) : \varphi \wedge \psi \text{ is big}\}.$$

It is easy to check that p is a complete type. In particular, we can write

$$[\varphi] = \{p\} \cup \bigcup_{\varphi \wedge \neg\psi \text{ small}} [\varphi \wedge \neg\psi],$$

which implies that $||[\varphi]|| \leq \kappa$, as $|L(A)| \leq \kappa$. Hence, we can build a tree of big formulas: $\varphi_\emptyset(x) = (x = x)$ split into $\varphi_0(x), \varphi_1(x)$ split into $\varphi_{00}(x), \varphi_{01}(x), \varphi_{10}(x), \varphi_{11}(x)$, etc. Let $B \subseteq A$ be the (countable) set consisting of all parameters of φ_η , for $\eta \in 2^{<\omega}$. Then we have $|S_1(B)| \geq 2^{\aleph_0}$, as for every path $\mu \in 2^\omega$ we can consider a partial type

$$p_\mu(x) = \{\varphi_{\mu|_n}(x) : n \in \mathbb{N}\}$$

and those are pairwise inconsistent with each other. Thus, we contradict ω -stability. $\square$

Lemma 10.8. If T is ω -stable and $A \subseteq \mathfrak{C}$ is a subset, then isolated types are dense in $S_1(A)$.

Proof. If φ is a consistent formula in one variable such that $[\varphi] \subseteq S_1(A)$ does not contain any isolated points, then we can split it into $[\varphi \wedge \psi]$ and $[\varphi \wedge \neg\psi]$ that both have no isolated points. Doing this inductively we can build a tree of formulas that would contradict ω -stability for the set of the parameters involved. $\square$

Definition 10.9. Fix a small set of parameters $A \subseteq \mathfrak{C}$. A model $A \subseteq M \prec \mathfrak{C}$ is called *prime over A*, if for any $A \subseteq N \prec \mathfrak{C}$ there is an elementary map $f : M \rightarrow N$ that restricts to the identity on A . Equivalently, M is a prime model of $\text{Diag}_{\text{el}}(A)$. We call M *atomic over A* if for any finite tuple $a \subseteq M$, $\text{tp}(a/A)$ is isolated.

Theorem 10.10. If T is ω -stable, then it has a prime model over any parameter set A . In particular, it has a unique up to isomorphism prime model.

Proof. We construct a sequence $(a_\alpha)_{\alpha < \gamma}$, for some ordinal γ that will arise from the construction, inductively so that $a_\alpha \in \mathfrak{C}$ realises an isolated type over $Aa_{<\alpha}$. Such a type always exists by Lemma 10.8. At some point we have to finish, in the sense that the new elements become just copies, because $\mathfrak{C}$ has some (very big but bounded) cardinality.

Let $A \subseteq N \prec \mathfrak{C}$ be any model containing A . We can construct an elementary (over A) embedding $A \cup \{a_\alpha : \alpha < \gamma\} \rightarrow N$ inductively, by taking realisations of the appropriate isolated types. In particular, we see that $|\gamma| < |N|$.

It remains to show that $M = A \cup \{a_\alpha : \alpha < \gamma\}$ is a model. If it is not a model, then it does not satisfy the Tarski-Vaught test, so there is a consistent formula $\varphi(x, a_\gamma)$ (possibly with hidden parameters from A) such that $\mathfrak{C} \models \exists x \varphi(x, a_\gamma)$. By Lemma 10.8 there is an isolated type p in $[\varphi(x, a_{<\gamma})]$, in particular we can define $a_{\gamma+1}$ as realising p . Since we know that our construction has stopped, this means that p must be isolated by $x = a_\beta$ for some $\beta < \gamma$. Hence M satisfies the Tarski-Vaught test, and thus is a model of T . $\square$

The model we have constructed is called a *constructible model (over A)*, i.e., it has a construction $M = A \cup \{a_\alpha : \alpha < \gamma\}$ where for every $\beta < \gamma$ the type $\text{tp}(a_\beta / Aa_{<\beta})$ is isolated. Ressayre proved that any two constructible models over A are isomorphic. However, in general a model that is prime over A does not have to be constructible over A . Shelah (1972) showed that, in case of an ω -stable theory, a prime model over any parameter set A is unique up to an isomorphism. See [5] for how one can deduce it from Ressayre's theorem. Uniqueness is not true in general - there are theories which admit two non-isomorphic models prime over some parameter set.

Note that by Lemma 10.3 every a constructible model M over A is atomic over A . In particular using the definition of a prime model we get the following.

Corollary 10.11. If there exists a constructible model M over A , then every model that is prime over A is also atomic over A .

Exercises, sheet 7.

1. Prove that an isolated type in a complete theory is realised.
2. Show that if a prime model exists, it is atomic.
3. Determine prime models of ACF_p , DLO_0 . Do these have prime models over parameter sets? What are those in the cases they exist?
4. Show that if T is a complete theory with Skolem functions, then the prime model over any parameter set A exists.
5. Show that for a complete theory T in a countable language L , T has a prime model if and only if for every n isolated types are dense in $S_n(\emptyset)$.
6. A theory T in a countable language L is called *small* if for every n , $S_n(\emptyset)$ is countable. Show that a small theory has a prime model.
7. Show that if T in a countable language L is not small, then it has $2^{\aleph_0}$ isomorphism classes of models of cardinality $\aleph_0$ (the maximal possible cardinality).

11. CANTOR-BENDIXSON, MORLEY, AND SHELAH RANKS

Let X be a (compact Hausdorff) topological space.

Definition 11.1. The *Cantor-Bendixson derivative* X' of X is the subspace of non-isolated points. We define inductively the iterated Cantor-Bendixson derivative by:

- $X^0 = X$,
- $X^{\alpha+1} = (X^\alpha)'$,
- $X^\beta = \bigcap_{\alpha < \beta} X^\alpha$ for limit β .

For $p \in X$ we denote by $\text{CB}(p)$ the smallest ordinal α (or ∞ in case it does not exists) such that $p \notin X^{\alpha+1}$. We write $\text{CB}(X)$ for the smallest ordinal α (or ∞ in case it does not exists) such that $X^{\alpha+1} = \emptyset$, and we declare $\text{CB}(\emptyset) = -1$.

Exercise 11.2. ⑥ Show that $\text{CB}(X) = \max\{\text{CB}(p) : p \in X\}$. Furthermore, if X is a Stone space, show that $\text{CB}(p) = \min\{\text{CB}(Y) : p \in Y, Y \text{ is clopen in } X\}$.

Let T be a complete L -theory and fix a monster model $\mathfrak{C}$ of T .

Definition 11.3. Let $\varphi(x, a)$ be an L -formula with parameters from $\mathfrak{C}$ and write $[\varphi(x, a)] \subseteq S_x(\mathfrak{C})$ for the set of global types containing $\varphi(x, a)$. The *Morley rank* of $\varphi(x, a)$ is $\text{CB}([\varphi(x, a)])$ and is denoted by $\text{MR}(\varphi(x, a))$, or by $\text{MR}(X)$ where $X = \varphi(\mathfrak{C}, a)$. If $p(x)$ is a partial global type, we denote by $\text{MR}(p)$ the minimum of Morley ranks of formulas $\varphi(x, a)$ such that $p(x) \vdash \varphi(x, a)$. In particular, if $p(x) \in S_x(A)$ for some small subset $A \subseteq \mathfrak{C}$, we treat p as a partial global type and thus $\text{MR}(p) = \min\{\text{MR}(\varphi(x, a)) : \varphi(x, a) \in p(x)\}$.

Lemma 11.4. Let X be an infinite compact Hausdorff space. Then there is an infinite sequence of disjoint open subsets $Y_1, Y_2, \dots \subseteq X$. If X is a Stone space, Y_i 's can be assumed to be clopen.

Proof. ⑥ Exercise. □

The following is often given as a definition of the Morley rank.

Lemma 11.5. Let $\varphi(x)$ be a formula with parameters from $\mathfrak{C}$. Then:

- $\text{MR}(\varphi) \geq 0$ if and only if $\varphi(x)$ is consistent;
- $\varphi(x) \geq \alpha + 1$ if and only if there are $\psi_1(x), \psi_2(x), \dots$ with parameters from $\mathfrak{C}$ such that $\psi_i \vdash \varphi$, that are pairwise inconsistent, and such that $\text{MR}(\psi_i) \geq \alpha$;
- $\text{MR}(\varphi) \geq \delta$ for a limit ordinal δ if and only if $\text{MR}(\varphi) \geq \alpha$ for all $\alpha < \delta$.

Proof. The first and the third bullets are easy so we focus on the second one. Note that $\text{CB}([\varphi]) \geq \alpha + 1$ means that $[\varphi]^{\alpha+1} \neq \emptyset$. This is equivalent to saying that $[\varphi]^\alpha$ has non-isolated points, or to saying that $[\varphi]^\alpha$ is infinite. If the right hand side of the second bullet is satisfied, then $[\varphi]^\alpha$ is infinite, because we can choose complete global types $p_i \in [\psi_i]$ with $\text{MR}(p_i) = \text{MR}(\psi_i)$, hence $p_i \in [\psi_i]^\alpha \subseteq [\varphi]^\alpha$.

On the other hand, if $[\varphi]^\alpha$ is infinite, we use Lemma 11.4 to get $[\psi'_i] \cap [\varphi]^\alpha = Y_i \subseteq X = [\varphi]^\alpha$ and $\psi_i = \varphi \wedge \psi'_i$ (modified inductively so that they are pairwise inconsistent) witness the validity of the right hand side of the second bullet. □

From the proof we get the following.

Corollary 11.6. $\text{MR}(\varphi) \geq \alpha + 1$ if and only if for any natural n there are pairwise inconsistent $\psi_1, \dots, \psi_n$, each $\psi_i \vdash \varphi$, such that $\text{MR}(\psi_i) \geq \alpha$ for $i = 1, \dots, n$.

Let $X \subseteq \mathfrak{C}^x$ be a subset definable over $\mathfrak{C}$. If $\text{MR}(X) = \alpha$ we define the *Morley degree* of X , denoted $\text{MD}(X)$, to be the biggest natural number n such that we can write X as a disjoint union of n definable (with parameters) sets $Y_1, \dots, Y_n$ with $\text{MR}(Y_i) = \alpha$. By the above corollary, such n exists. We also define

$$\text{MD}(p) = \min \left\{ \text{MD} \left(\bigwedge p_0(x) \right) : \text{finite } p_0(x) \subseteq p(x), \text{MR}(p) = \text{MR} \left(\bigwedge p_0(x) \right) \right\},$$

for a partial type p with $\text{MR}(p) < \infty$.

Exercise 11.7. ⑥ Show that $\text{MR}(\varphi(x, a))$ does not depend on whether we calculate it in $\mathfrak{C}$ or in a bigger monster model $\mathfrak{C}' \prec \mathfrak{C}$.

Theorem 11.8. Assume that the language L is countable. A complete L -theory T is ω -stable if and only if for every finite tuple of variables x we have $\text{MR}(x = x) < \infty$.

Proof. Assume that $\text{MR}(x = x) < \infty$ for any finite tuple of variables x . Let A be a countable set of parameters in $\mathfrak{C}$. For every $p \in S_x(A)$ choose a formula $\varphi(x, a)$ with parameters from A , such that $\text{MR}(p) = \text{MR}(\varphi(x, a))$. We claim that this is a finite-to-one map from $S_x(A)$ to the set $F_x(A)$ of formulas in variable x with A -parameters. Indeed, if there is $\varphi(x, a)$ such that there are infinitely many types $p_1, p_2, \dots \in [\varphi(x, a)] \subseteq S_x(A)$ of the same $\text{MR}(p_i) = \text{MR}(\varphi(x, a)) = \alpha$, then for every natural n , $[\varphi(x, a)] \subseteq S_x(A)$ can be split into n disjoint clopens of $\text{MR} \geq \alpha$, which implies $\text{MR}(\varphi(x, a)) \geq \alpha + 1$ yielding a contradiction. In particular, $|S_x(A)| \leq \aleph_0 \cdot |F_x(A)| = \aleph_0$.

On the other hand, assume that $\text{MR}(x = x) = \infty$ for some finite tuple of variables x . Note that the image of the Morley rank is of the form $[0, \alpha] \cup \{\infty\}$ for some $\alpha \leq (2^{|T|})^+ \oplus$. In particular, $\text{MR}(\varphi) = \infty \iff \text{MR}(\varphi) \geq \alpha + 1$. Hence, for any $\varphi(x)$ with $\text{MR}(\varphi) = \infty$, we can find $\psi(x)$ such that $\text{MR}(\varphi \wedge \psi) = \text{MR}(\varphi \wedge \neg\psi) = \infty$. By iterating we can build a tree of formulas $(\varphi_\eta)_{\eta \in 2^{<\omega}}$ with $\text{MR}(\varphi_\eta) = \infty$. This violates stability for the space of types over parameters appearing in φ_η 's. $\square$

That is why a complete theory T (in any language L) is called *totally transcendental*, if $\text{MR}(x = x) < \infty$ for all finite tuples of variables x .

Definition 11.9. An infinite definable set $X \subseteq \mathfrak{C}^x$ is called *strongly minimal* if for every definable set $Y \subseteq \mathfrak{C}^x$ one of $X \cap Y, X \cap Y^c$ is finite.

Exercise 11.10. $\oplus$ Show that:

- $\text{MR}(X) = 0 \iff X$ is finite.
- $\text{MR}(X \cup Y) = \max(\text{MR}(X), \text{MR}(Y))$.
- For $\sigma \in \text{Aut}(\mathfrak{C})$, $\text{MR}(\sigma(X)) = \text{MR}(X)$.
- X is strongly minimal $\iff \text{MR}(X) = 1, \text{MD}(X) = 1$.

In order to prove that in an ω -stable theory prime models over parameter sets are unique up to isomorphism, we will need a related rank introduced by Morley and used by Shelah for this purpose.

Definition 11.11. For a formula $\varphi(x, a)$ we define its *Shelah two-rank*, denoted by $\text{TR}(\varphi(x, a))$ and ordinal or ∞ valued, inductively by the following:

- $\text{TR}(\varphi(x, a)) = -1 \iff \varphi(\mathfrak{C}, a) = \emptyset$.
- $\text{TR}(\varphi(x, a)) \geq \alpha + 1$ if and only if there is a formula $\psi(x, b)$ such that $\text{TR}(\varphi(x, a) \wedge \psi(x, b)) \geq \alpha$ & $\text{TR}(\varphi(x, a) \wedge \neg\psi(x, b)) \geq \alpha$.
- $\text{TR}(\varphi(x, a)) \geq \delta$ for a limit ordinal δ if and only if $\text{TR}(\varphi(x, a)) \geq \alpha$ for all $\alpha < \delta$.
- $\text{TR}(\varphi(x, a)) = \infty$ if for all ordinals α , $\text{TR}(\varphi(x, a)) \geq \alpha$.

For a partial type p we define

$$\text{TR}(p) = \min \left\{ \text{TR} \left(\bigwedge p_0(x) \right) : \text{finite } p_0(x) \subseteq p(x) \right\}.$$

Exercise 11.12. $\oplus$ Show (similarly as for MR) that for a complete theory T in a countable language, T is ω -stable if and only if for all finite tuples x of variables $\text{TR}(x = x) < \infty$.

Lemma 11.13. Let T be an ω -stable complete theory T in a countable language L . The Shelah two-rank has the following properties:

- (1) The range of TR is of the form $[0, \alpha_0]$ for some ordinal $\alpha_0 < (2^{\aleph_0})^+$.
- (2) TR is invariant under automorphisms of definable sets.

- (3) A type $p \in S(A)$ of minimal TR in $[\varphi(x, a)] \subseteq S(A)$ is isolated over A (this is how we showed that isolated types are dense over any parameter sets in ω -stable theories).
- (4) If $p \subseteq q$, then $\text{TR}(p) \geq \text{TR}(q)$.
- (5) If p is a partial type over A , there is at most one $q \in S(A)$ with $p \subseteq q$ and $\text{TR}(p) = \text{TR}(q)$.

Proof. © Exercise. □

12. INDISCERNIBLES IN STABLE THEORIES

Let T be a complete stable theory with a monster model $\mathfrak{C}$.

Theorem 12.1. Let $(a_i)_{i \in I}$ be an indiscernible sequence for an infinite linear order I . Then $\{a_i : i \in I\}$ is an indiscernible set.

Proof. Let κ be such that T is κ -stable. Assume for a contradiction that $\{a_i : i \in I\}$ is not an indiscernible set. Then there is a formula $\varphi(x_1, \dots, x_n)$ such that

$$\models \varphi(a_{i_1}, \dots, a_{i_n}) \wedge \neg \varphi(a_{i'_1}, \dots, a_{i'_n})$$

for some permutation $(i'_1, \dots, i'_n) = (i_{\sigma(1)}, \dots, i_{\sigma(n)})$. Choose a dense linear order $(I', <)$ where each non-empty interval is of cardinality bigger than κ with a dense subset $J \subseteq I'$ of cardinality κ (prove that such exists ©). Consider

$$p(x_i : i \in I') = \bigcup_{i_1 < \dots < i_n \in I'} \text{tp}(a_1 < \dots < a_n)(x_{i_1}, \dots, x_{i_n}),$$

where by $a_1 < \dots < a_n$ we mean any tuple of n elements in the sequence whose indexes in I are increasing. Then p is a type containing the relevant information about φ , so without loss of generality we can assume that $I = I'$.

By writing $\sigma = \tau_1 \circ \dots \circ \tau_m$ as a composition of transpositions of the form $(l, l+1)$ there is some $1 \leq t < m$ such that

$$\models \varphi(a_{i_{\sigma_t(1)}}, \dots, a_{i_{\sigma_t(n)}}) \leftrightarrow \neg \varphi(a_{i_{\sigma_{t+1}(1)}}, \dots, a_{i_{\sigma_{t+1}(n)}}),$$

where $\sigma_t = \tau_1 \circ \dots \circ \tau_t$. By writing

$$\chi(a_{i_1}, \dots, a_{i_n}) \leftrightarrow \varphi(a_{i_{\sigma_t(1)}}, \dots, a_{i_{\sigma_t(n)}})$$

with $i_1 < \dots < i_n \in I$, and by assuming that $\tau_{t+1} = (j, j+1)$, we get that for

$$\delta(x, y) = \chi(a_{i_1}, \dots, a_{i_{j-1}}, x, y, a_{i_{j+2}}, \dots, a_{i_n}),$$

we have

$$\models \delta(a_{i_j}, a_{i_{j+1}}) \wedge \neg \delta(a_{i_{j+1}}, a_{i_j}).$$

Let $A = \{a_j : j \in J\}$ so that $|A| \leq \kappa$. We will show that $|S(A)| > \kappa$. Consider $i, i' \in I$ such that $i_{j-1} < i < i' < i_{j+2}$. We claim that

$$\text{tp}(a_i/A) \neq \text{tp}(a_{i'}/A).$$

Indeed, choose $r \in J$ such that $i < r < i'$. By indiscernibility of the sequence we have $\models \delta(a_i, a_r) \wedge \neg \delta(a_{i'}, a_r)$ which finishes the proof. □

Theorem 12.2. Fix a formula $\varphi(x, y)$. Then there is a natural number $r = r_{\varphi(x, y)}$ such that for all indiscernible sets $\{a_i : i \in I\}$ and all b we have

$$|\{i \in I : \models \varphi(a_i, b)\}| \leq r \text{ or } |\{i \in I : \models \neg \varphi(a_i, b)\}| \leq r.$$

Proof. Assume such r does not exist. Then, by compactness, there is an indiscernible set $(a_n)_{n \in \mathbb{N}}$ and b such that both $A = \{n \in \mathbb{N} : \models \varphi(a_n, b)\}$ and its complement $B = \{n \in \mathbb{N} : \models \neg \varphi(a_n, b)\}$ are infinite. In particular, the truth value of $\varphi(a_n, b)$ alternates infinitely often.

Consider the variables $(x_i)_{i < \kappa}, (y_J)_{J \subseteq \kappa}$ and the collection

$$p(x, y) = \{\varphi(x_i, y_J) : i \in J \subseteq \kappa\} \cup \{\neg \varphi(x_i, y_J) : i \notin J \subseteq \kappa\}.$$

Given a finite subset of p mentioning only k elements $i_1, \dots, i_k$ (both in indexes i and in subsets J considered). Rename the variables so that these are $1, \dots, k$. For any $J \subseteq \{1, \dots, k\}$ find $n_1 < \dots < n_k$ in $\mathbb{N}$ such that $n_i \in A \iff i \in J$. Let $\sigma_J \in \text{Aut}(\mathfrak{C})$ be an automorphism that maps $a_{n_1}, \dots, a_{n_k}$ to $a_1, \dots, a_k$. If $b_J := \sigma_J(b)$, we get that

$$i \in J \iff \varphi(a_{n_i}, b) \iff \models \varphi(a_i, b_J).$$

This shows that $p(x, y)$ is a type. Let $((a_i)_{i < \kappa}, (b_J)_{J \subseteq \kappa}) \models p$ and let $C = \{a_i : i < \kappa\}$. Then $|C| \leq \kappa$ but $|S(C)| \geq 2^\kappa$ which shows that T is not κ -stable. Since κ was arbitrary we get that T is not stable, which finishes the proof. $\square$

Definition 12.3. Let $I \subseteq \mathfrak{C}$ be an infinite indiscernible set over A , and let $B \supseteq A$. Define

$$\text{Av}(I/B) = \{\varphi(x, b) \in L(B) : \mathfrak{C} \models \varphi(a, b) \text{ for all but finitely many } a \in I\}.$$

We call $\text{Av}(I/B)$ the *average type* of I over B . The fact that it is a complete type follows from Theorem 12.2 (e).

Proposition 12.4 (Average types are definable). Let $I \subseteq \mathfrak{C}$ be an indiscernible set over A , and $\varphi(x, y)$ be an L -formula, where I is of sort x . Let $p = \text{Av}(I/\mathfrak{C})$ be the average type of I over $\mathfrak{C}$. There is a formula $\psi(y)$ using parameters only from $A \cup I$, such that for all $b \in \mathfrak{C}$ of sort y ,

$$\varphi(x, b) \in p \iff \models \psi(b).$$

Proof. This is by taking $r = r_\varphi$ as in Theorem 12.2 and testing $\varphi(x, b)$ on a $2r + 1$ tuple of elements of I (e). $\square$

Corollary 12.5. Assume that the language L is countable. Let $I \subseteq \mathfrak{C}$ be an indiscernible set over A , $A \subseteq B$, and let $p = \text{Av}(I/B)$. Then

$$|\{a \in I : a \text{ does not realise } \text{Av}(I/B)\}| \leq |B| + \aleph_0.$$

Proof. Exercise (e). $\square$

Exercises, sheet 8. Let T be a complete theory in a countable language L , with a monster model $\mathfrak{C}$. Unless stated otherwise, all parameter sets and sequences are small. Definable sets may use parameters from $\mathfrak{C}$. In Exercises 3,4 assume that T is ω -stable, and in Exercises 6,7,8 assume that T is stable.

1. Show that $\text{MR}(X) = 0$ if and only if X is non-empty and finite, and that in this case $\text{MD}(X) = |X|$.
2. Let X be a definable set of cardinality m , where $1 \leq m < \omega$. Prove that

$$\text{TR}(X) = \lfloor \log_2 m \rfloor.$$

Show that if X is strongly minimal, then $\text{TR}(X) = \omega$. Here $\text{TR}(X)$ denotes the two-rank of a formula defining X .

3. Prove the following properties of the two-rank.

- (i) If $p(x)$ is a consistent partial type over $A \subseteq B$, there is at most one $q \in S_x(B)$ such that $p \subseteq q$ and $\text{TR}(q) = \text{TR}(p)$.
- (ii) For every $p \in S_x(A)$ there is a finite subset $A_0 \subseteq A$ such that

$$\text{TR}(p|_{A_0}) = \text{TR}(p).$$

Deduce that p is the unique extension of $p|_{A_0}$ to a complete type over A with the same two-rank.

- (iii) Let $\varphi(x, a)$ be a consistent formula over A . Show that a type of minimal two-rank in $[\varphi(x, a)] \subseteq S_x(A)$ is isolated. Conclude that isolated types are dense in $S_x(A)$.
- 4. Let T be the theory of an equivalence relation E with exactly two classes, both infinite, and let $a \in \mathfrak{C}$. Show that there is a unique type $p \in S_1(\emptyset)$, and find all its extensions to complete types over $\{a\}$. Compute the Morley rank and the two-rank of p and of each extension. Deduce that preserving Morley rank does not imply uniqueness of an extension, and that an extension preserving two-rank need not exist.
- 5. Solve Exercise 11.12.
- 6. Let $I \subseteq \mathfrak{C}$ be an infinite indiscernible set over A , and let $B \supseteq A$.
 - (i) Use Theorem 12.2 to show that $\text{Av}(I/B)$ is a complete type over B . Show that it is finitely satisfiable in I , meaning that every finite subset is realised by an element of I .
 - (ii) Show that for every infinite subset $J \subseteq I$,

$$\text{Av}(J/B) = \text{Av}(I/B).$$

- (iii) If c realises $\text{Av}(I/A \cup I)$, show that $c \notin I$ and that $I \cup \{c\}$ is indiscernible over A .
- 7. Let I be as in the previous exercise, and put $p = \text{Av}(I/\mathfrak{C})$. Fix an L -formula $\varphi(x, y)$ and let $r = r_\varphi$ be given by Theorem 12.2. For any distinct $a_1, \dots, a_{2r+1} \in I$, find an explicit formula $\psi(y, a_1, \dots, a_{2r+1})$ such that for every $b \in \mathfrak{C}^y$,

$$\varphi(x, b) \in p \iff \mathfrak{C} \models \psi(b, a_1, \dots, a_{2r+1}).$$

Deduce that for every fixed countably infinite $J \subseteq I$, the type p is *definable over J* : for every $\varphi(x, y)$, the set $\{b : \varphi(x, b) \in p\}$ is definable with parameters from J .

- 8. Let I and B be as in Exercise 6. Show that

$$|\{a \in I : a \text{ does not realise } \text{Av}(I/B)\}| \leq |B| + \aleph_0.$$

Deduce that if $|I| > |B| + \aleph_0$, then I contains $|I|$ realisations of $\text{Av}(I/B)$.

13. UNIQUENESS OF PRIME MODELS IN ω -STABLE THEORIES

Fix a complete ω -stable theory T with a monster model $\mathfrak{C}$ for the rest of this section. From now on, we begin using model theoretic notation for parameter sets, i.e., we will write $ABa_{<\alpha}$ for $A \cup B \cup \{a_k : k < \alpha\}$ (and similarly for other unions of tuples). We may also use several properties from Section 10.

Lemma 13.1. Let $\{a_i : i < \alpha\}$ be a sequence indexed by an infinite ordinal α and let $p_i = \text{tp}(a_i/Aa_{<i})$. If $p_0 \subseteq p_i$ and $\text{TR}(p_0) = \text{TR}(p_i)$ for all $i < \alpha$, then $\{a_i : i < \alpha\}$ is indiscernible over A .

Proof. We will show inductively, that for all natural k , for any $i_1 < \dots < i_k$ we have

$$\text{tp}(a_{i_1} \dots a_{i_k}/A) = \text{tp}(a_1, \dots, a_k/A).$$

By the assumption all a_i 's have the same type over A . For the induction step fix $i_1 < \dots < i_k < i_{k+1}$ and look at $p = \text{tp}(a_{i_{k+1}}/Aa_{i_1} \dots a_{i_k})$. By the inductive hypothesis there is an automorphism $\sigma \in \text{Aut}(\mathfrak{C}/A)$ such that $\sigma(a_{i_j}) = a_j$ for $j = 1, \dots, k$. Consider $q = \sigma(p) = \text{tp}(\sigma(a_{i_{k+1}})/Aa_{<k})$ and note that $p_0 \subseteq q$. Since Shelah's two-rank is preserved under automorphism, we get that

$$\text{TR}(p_0) \geq \text{TR}(q) = \text{TR}(p) \geq \text{TR}(p_{i_{k+1}}) = \text{TR}(p_0).$$

Thus $\text{TR}(p_0) = \text{TR}(q) = \text{TR}(p_k)$ and $p_0 \subseteq p_k, q$, from which we conclude that $p_k = q$, as there can be only one extension to a complete type preserving TR. This finishes the inductive step and gives us an indiscernible set by Theorem 12.1. $\square$

Lemma 13.2. Let $I = \{a_i : i < \aleph_1\}$ be an indiscernible set over A and let B be a countable set. Then there exists an increasing $\beta : \aleph_1 \rightarrow \aleph_1$ such that $\{a_{\beta_j} : j < \aleph_1\}$ is indiscernible over AB .

Proof. Let $p = \text{Av}(I/ABI)$ and choose $\alpha_0 < \aleph_1$ such that $\text{TR}(p) = \text{TR}(\text{Av}(I/ABa_{<\alpha_0}))$. Note that by Corollary 12.5, for every $\beta < \aleph_1$ there is $\beta < \beta' < \aleph_1$ such that $a_i \models \text{Av}(I/ABa_{<\beta})$ for $i \geq \beta'$. Hence, inductively we can find an increasing sequence $(\beta_j)_{j < \aleph_1}$ such that

$$a_{\beta_j} \models \text{Av}(I/ABa_{<\alpha_0} a_{\beta_{<j}}).$$

Assume that $\beta_0 > \alpha_0$. Then we can use Lemma 13.1 to get the conclusion. $\square$

Lemma 13.3. Assume that $I = \{a_i : i < \alpha\}$ is indiscernible over A with $\alpha \geq \omega$ and let $B = A \cup \{a_i : i < \alpha\}$. Then $\{a_i : i < \alpha\}$ is a maximal indiscernible set over A , in any prime model M over B .

Proof. Let $p = \text{Av}(I/B)$. It is not isolated as for any $\varphi(x, b) \in p(x)$, the formulas $\varphi(x, b) \wedge (x = a_i)$ are consistent for infinitely many i 's. Hence $p(x)$ cannot be realised in any model M prime over B . This proves the lemma, as for an element a we have

$$\{a_i : i < \alpha\} \cup \{a\} \text{ is indiscernible over } A \iff a \models p.$$

$\square$

Proposition 13.4. Let M be a constructible model over A . Then there is no uncountable indiscernible set over A in M .

Proof. Assume for a contradiction that $I = \{b'_i : i < \aleph_1\}$ is an uncountable indiscernible set over A in M . Let $p = \text{Av}(I/M) \in S_x(M)$. Choose a tuple $c_0 \in M$ and $\psi(x, c_0) \in p(x)$ such that $\text{TR}(p) = \text{TR}(\psi(x, c_0))$. If $M = A \cup \{a_i : i < \alpha\}$ is a construction of M over A , then $M = Ac_0 \cup \{a_i : i < \alpha\}$ is a construction of M over Ac_0 . Furthermore, by Lemma 13.2, we can pass to a subsequence $\{b_i : i < \aleph_1\}$ that is indiscernible over Ac_0 . Therefore, without loss of generality $c_0 \in A$ and $\{b_i : i < \aleph_1\}$ is indiscernible over A .

Let $B = A \cup \{b_i : i < \omega\}$, N be a prime model over B (for example a constructible model over B), and $f : M \rightarrow N$ be an elementary embedding over A with $b^i := f(b_i)$ for $i < \aleph_1$. Then $\{b^i : i < \aleph_1\}$ is indiscernible over A and by Lemma 13.2, after potentially passing to a subsequence, we can assume that $\{b^i : \beta \leq i < \aleph_1\}$ is indiscernible over B . Denote by $q = \text{tp}(b^\beta/B)$. Then $q \neq p|_B$ as otherwise

$\{b_i : i < \omega\} \cup \{b^\beta\}$ would be indiscernible over A , which contradicts Lemma 13.3. However, note that

$$\text{TR}(\psi(x, c_0)) \geq \text{TR}(p|_B) \geq \text{TR}(p) = \text{TR}(\psi(x, c_0)),$$

so we get that $\text{TR}(q) < \text{TR}(p|_B) = \text{TR}(p)$ as they both extend $\text{TR}(p|_A)$. Let $\theta(x, c_1) \in q$ be a formula that satisfies $\text{TR}(\theta(x, c_1)) = \text{TR}(q)$. Let $r = r_\theta$ be the number from Theorem 12.2 and let $\chi(x, y_1, \dots, y_r)$ be the $L(A)$ -formula isolating the type of $c_1, b^{\beta+1}, \dots, b^{\beta+r}$ over A . Denote by $c_2 \in M$ an element such that

$$M \models \chi(c_2, b_{\beta+1}, \dots, b_{\beta+r}).$$

Such exists because $N \models \exists x \chi(x, b^{\beta+1}, \dots, b^{\beta+r})$ and $b^{\beta+1} \dots b^{\beta+r} \equiv_A b_{\beta+1} \dots b_{\beta+r}$. As $N \models \theta(b^{\beta+i}, c_1)$ for $i = 0, \dots, r$, we get that $M \models \theta(b_{\beta+i}, c_2)$ for $i = 0, \dots, r$. Hence

$$|\{i < \aleph_1 : M \models \theta(b_i, c_2)\}| > r,$$

which implies that $\theta(x, c_2) \in p$. This gives a contradiction as

$$\text{TR}(p) \leq \text{TR}(\theta(x, c_2)) = \text{TR}(\theta(x, c_1)) < \text{TR}(p).$$

□

We immediately ⑥ get the following.

Corollary 13.5. Let M be a model that is prime over A . Then there is no uncountable indiscernible set over A in M .

Lemma 13.6. Suppose that M is atomic over B and let $p_i \in S(B)$ for $i < \alpha$. Consider

$$B' = B \cup \bigcup_{i < \alpha} p_i(M).$$

Then M is atomic over B' .

Proof. Let c be a tuple from M and let $p = \text{tp}(c/B')$. Take $\varphi(x, b) \in p$, $b \in B'$, such that $\text{TR}(p) = \text{TR}(\varphi(x, b))$. Let $\psi(x, b)$ be an $L(Bb)$ -formula isolating $p|_{Bb}$. Let

$$\chi(x, b) = \varphi(x, b) \wedge \psi(x, b).$$

We claim that $\chi(x, b)$ isolates p . If that were not the case, there would be a formula $\chi_1(x, b_1) \in p$ with parameters $b_1 \in B'$ such that

$$\models \exists x \chi(x, b) \wedge \neg \chi_1(x, b_1).$$

Let $d \in M$ be such that $M \models \chi(d, b) \wedge \neg \chi_1(d, b_1)$. Because $\text{tp}(db_1/Bb)$ is isolated, there is b_2 such that $db_1 \equiv_{Bb} cb_2$. In particular, then $\chi(c, b) \wedge \neg \chi_1(c, b_2)$ holds. Since $b_1 \in B'$ and $b_2 \equiv_B b_1$, by the assumption we get that $b_2 \in B'$. Now we make some observations about the two-ranks:

- $\text{TR}(\varphi(x, b)) \geq \text{TR}(\chi(x, b)) \geq \text{TR}(\chi(x, b) \wedge \chi_1(x, b_1)) \geq \text{TR}(p) = \text{TR}(\varphi(x, b))$, so they are all equal.
- In particular, by definition $\text{TR}(\chi(x, b) \wedge \neg \chi_1(x, b_1)) < \text{TR}(\chi(x, b))$.
- $\text{TR}(\chi(x, b) \wedge \neg \chi_1(x, b_1)) = \text{TR}(\chi(x, b) \wedge \neg \chi_1(x, b_2))$ as $bb_1 \equiv_B bb_2$.
- $\text{TR}(\chi(x, b) \wedge \neg \chi_1(x, b_2)) \geq \text{TR}(p) = \text{TR}(\chi(x, b))$ as $M \models \chi(c, b) \wedge \neg \chi_1(c, b_2)$ and $bb_2 \subseteq B'$.

Together those give a contradiction. □

Theorem 13.7 (Characterisation of prime models over sets for ω -stable theories). Suppose that $A \subseteq M, B \subseteq N$ for $M, N \models T$ and let $F : A \rightarrow B$ be an elementary bijection. Assume that:

- (1) M is atomic over A and N is atomic over B .
- (2) In M (respectively N) there is no uncountable indiscernible set over A (respectively over B).

Then F can be extended to an isomorphism between M and N .

Proof. We prove the following statement using induction based on an ordinal α :

($\star$) If $c \in A$ and $\text{TR}(\varphi(x, c)) = \alpha$, then we can extend F to an elementary bijection $F' : A' \rightarrow B'$, where

$$A' = A \cup \{b \in M : M \models \varphi(b, c)\} \text{ and } B' = B \cup \{d \in N : N \models \varphi(d, F(c))\}.$$

This is enough to conclude, as at some point we can take $\varphi(x) = (x = x)$.

Suppose that ($\star$) holds for every $\beta < \alpha$ (this is an empty condition when α is zero). We start by constructing a sequence in M inductively: let a_i be any element in M that realises over $A_{a_{<i}}$ any type p_i such that $\varphi(x, c) \in p_i$ and $\text{TR}(p_i) = \text{TR}(\varphi(x, c))$. Assume we have defined $a_i \in M$, for $i < i_0$ where i_0 is some ordinal, and that the construction stops at i_0 . If $i_0 \geq \omega$, then by Lemma 13.1 the set $\{a_i : i < i_0\}$ is indiscernible. Hence, by Proposition 13.4, $i_0 < \aleph_1$. Choose an embedding $j : i_0 \rightarrow \omega$ onto an initial interval and rename $a_n := a_i$ if $n = j(i)$. Let $\alpha_0 = j(i_0)$. Note that then (by indiscernibility) it is still the case that

$$\varphi(x, c) \in p_n \text{ and } \text{TR}(p_n) = \text{TR}(\varphi(x, c)) \text{ for } n < \alpha_0,$$

and there is no choice of $a_{\alpha_0} \in M$ that has these properties. Similarly, we construct a sequence $(b_n)_{n < \beta_0}$ in N , for some $\beta_0 \leq \omega$, such that

$$N \models \varphi(b_n, F(c)) \text{ and } \text{TR}(\text{tp}(b_n/B_{b_{<n}})) = \text{TR}(\varphi(x, F(c))) \text{ for } n < \beta_0,$$

and there is no choice of $b_{\beta_0} \in N$ that has these properties.

To construct F' with the desired properties we construct inductively an increasing sequence of elementary bijections $F_n : A_n \rightarrow B_n$, for $n < \omega$ with $A_n \subseteq M, B_n \subseteq N$, such that the following holds:

($\star\star$) M is atomic over A_n and N is atomic over B_n .

After the construction is finished we will set $F' = \bigcup_{n < \omega} F_n$. Now we describe how F_{n+1} is defined if we already have F_n . This is split into the following cases:

- $F_0 = F$.
- $n = 3m + 1$: if $m \geq \alpha_0$, then we define $F_{n+1} = F_n$. Otherwise, if $m < \alpha_0$ and $a_m \in M$ is isolated by $\psi(x, c)$ over A_n , then we choose $a'_m \in N$ such that $N \models \psi(a'_m, F_n(c))$ and declare $F_{n+1} = F_n \cup \{(a_m, a'_m)\}$.
- $n = 3m + 2$: this is a symmetric step where β_0 plays the role of α_0 and b_m plays the role of a_m .
- $n = 3m + 3$ (if $\alpha = 0$ we put $F_{n+1} = F_n$): enumerate types $p_i \in S_1(A_n)$, $i < i_0$, that are realized in M , satisfy $\varphi(x, c) \in p_i$, and $\text{TR}(p_i) < \alpha$. By ($\star\star$) each p_i is isolated, say by $\psi_i(x, c_i)$, and without loss of generality

$$\psi_i(x, c_i) \vdash \varphi(x, c) \text{ \& } \text{TR}(\psi_i(x, c_i)) = \text{TR}(p_i) < \alpha.$$

We define F_n^i , for $i < i_0$, inductively so that

$$\text{Dom}(F_n^i) = A_n \cup \bigcup_{j < i} \psi_j(M, c_j), \quad \text{Rng}(F_n^i) = B_n \cup \bigcup_{j < i} \psi_j(N, F_n^i(c_j)).$$

We start with $F_n^0 := F_n$, on the limit steps we take unions, and on the successor step we use the inductive assumption $(\star)$. We are allowed to do this as $\text{TR}(\psi_i(x, c_i)) < \alpha$, and M and N are atomic over those domain and codomain respectively, by Lemma 13.6 (and they do not have uncountable indiscernible sets over the domain and the codomain of F_n^i respectively). We put $F_{n+1} = \bigcup_{i < i_0} F_n^i$.

Together we get $F' = \bigcup_{n < \omega} F_n$. Clearly F' is an elementary function, so to finish the proof we need to check that A' is the domain of F' and B' is its image. By the construction it is enough to make sure that $A' \subseteq \text{Dom}(F')$, $B' \subseteq \text{Rng}(F')$. We check it for A' (the proof for B' is similar). Take $a \in M$ such that $M \models \varphi(a, c)$ and denote by $p = \text{tp}(a/Aa_{<\alpha_0})$. By the definition of α_0 , $\text{TR}(p) < \alpha$, so for some natural number $m < \alpha_0 \leq \omega$, $\text{TR}(p|_{Aa_{<m}}) < \alpha$. In particular, $\text{TR}(p|_{A_{3m}}) < \alpha$. This means that $\{a' \in M : a' \text{ realizes } p|_{A_{3m}}\} \subseteq A_{3m+1}$ so

$$a \in A_{3m+1} = \text{Dom}(F_{3m+1}) \subseteq \text{Dom}(F').$$

□

Corollary 13.8. In an ω -stable complete theory T prime models over a parameter set A exist and are unique up to an isomorphism over A . Furthermore, $M \models T$ is prime over A if and only if it is atomic over A and has no uncountable indiscernible set over A .

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