Exercises, sheet 0, differential equations. Let $\mathbb{Q} \subseteq R$ be a ring and let $\delta : R \to R$ be a derivation on R, i.e., for all $x, y \in R$ we have $\delta(x+y) = \delta x + \delta y, \delta(xy) = x\delta y + y\delta x$. Such a pair is called a (characteristic zero) *differential ring.* If $(R, \delta), (S, \delta')$ are two differential rings, a morphism between them is a homomorphism $f : R \to S$ such that $f(\delta x) = \delta' f(x)$. We denote by $R\{X\}$ the differential ring $R[X, X', X'', \dots]$, where δ is extended to $R\{X\}$ by $\delta(X^{(n)}) = X^{(n+1)}$. Let $f \in R\{X\}$. The *order* of f is the maximal n such that $X^{(n)}$ appears in f (and $\operatorname{ord}(r) = -\infty$ if $r \in R$). The *degree* of f is the maximal d such that $(X^{(n)})^d$ appears in f, where n is its order. Then we can write

$$f = \sum_{i=0}^{d} a_i \cdot (X^{(n)})^i$$

and $i_f := a_d \in R\{X\}$ is called the *initial* of f. We write $f \ll g$ if $\operatorname{ord}(f) < \operatorname{ord}(g)$ or $\operatorname{ord}(f) = \operatorname{ord}(g)$ and $\deg(f) < \deg(g)$.

1. Write a formula for the extended δ. Prove that for every morphism of differential rings $f : R \to S$ and every $s \in S$ there is a unique morphism of differential rings $\tilde{f} : R\{X\} \to S$ that extends f and maps X to s.
2. If $\operatorname{ord}(f) \geq 0$, we call $s_f = \frac{\partial f}{\partial X^{(n)}} \in R\{X\}$ the *separant* of f. Show that $s_f = s_{f'}$.
3. Denote by $\langle f \rangle = (f, f', f'', \dots) \subseteq R\{X\}$ the *differential ideal* (i.e., an ideal closed under δ) generated by f. Assume that $\operatorname{ord}(f) \geq 0$. Show that for any g there exists a natural number m and $r \in R\{X\}$ with $\operatorname{ord}(r) \leq \operatorname{ord}(f)$ such that
$$s_f^m g = r \mod \langle f \rangle.$$
4. Assume that $\operatorname{ord}(f) = n \geq 0$. Show that for any g there exist natural numbers m, l and $r \in R\{X\}$ with $r \ll f$ such that
$$i_f^l s_f^m g = r \mod \langle f \rangle.$$
Hint: use the usual division algorithm in the ring $R[X, X', \dots, X^{(n-1)}][X^{(n)}]$.
5. Let (k, δ) be a differentiable field of characteristic zero. Show that $f = (X'')^2 - 2X' \in k\{X\}$ is irreducible, however $\langle f \rangle$ is not a prime ideal.
6. Show that if $f \in k\{X\}$ is irreducible, then
$$I(f) = \{g \in k\{X\} : \exists m \geq 0, s_f^m \cdot g \in \langle f \rangle\}$$
is a prime differential ideal.
7. Show that all prime differential ideals in $k\{X\}$ are of this form.
8. Consider $f, g \in k\{X\}$ such that $\operatorname{ord}(f) > \operatorname{ord}(g) \geq 0$. Show that there is a differential field extension $k \subseteq l$ such that for some $x \in l$ we have $f(x) = 0$ and $g(x) \neq 0$.
9. We call (k, δ) *differentially closed*, if for all f, g as in the previous exercise, an x can be found already in k. Show that k can be embedded in a differentially closed differential field.
10. Write first-order axioms for differentially closed fields of characteristic zero.
11. Let $\mathcal{K}$ be the ring of (dense open) germs of meromorphic functions defined on dense open subsets of $\mathbb{C}$. The dense open subsets we are talking about are not required to be connected. Equip $\mathcal{K}$ with a derivative $\frac{d}{dz}$ and show that every ring ideal of $\mathcal{K}$ is automatically a differential ideal.

Later we will show that for every maximal ideal $\mathfrak{m}$ of $\mathcal{K}$, the quotient $\mathcal{K}/\mathfrak{m}$ is a differentially closed field extending $(\mathbb{C}(z), \frac{d}{dz})$. Furthermore, the isomorphism type of the differential field $\mathcal{K}/\mathfrak{m}$ does not depend on $\mathfrak{m}$.