Exercises, sheet 1.

1. Fill the gaps left in the proofs above. Exercises throughout the lecture notes are marked with the symbol ⓔ. Read the proof of completeness theorem carefully.
2. Prove that if X is a set of formulas and $X \vdash \varphi(c)$, where c is a constant not appearing in any formula from X, then $X \vdash \forall x\ \varphi(x)$.
3. Show that in any language for any formula φ there is a prenex normal form formula ξ such that $\vdash \varphi \leftrightarrow \xi$.
4. Show that isomorphic structures satisfy the same sentences. Hint: induction based on the complexity of the formula.
5. Let $L = (+, \cdot, <, (c_r)_{r\in\mathbb{R}}, \dots)$, where in (...) we include a unary (i.e., of arity one) function symbol for every function $f : \mathbb{R} \to \mathbb{R}$. Consider $\mathbb{R}$ as an L-structure. Show that there is a model M of $T = Th(\mathbb{R}) = \{\sigma : \mathbb{R} \models \sigma\}$ such that in M there exists $\varepsilon > 0$ with $\varepsilon < c_r^M$ for all $r \in \mathbb{R}_{>0}$. Hint: use compactness theorem for some $L' = L \cup \{\epsilon\}$-theory and put $\varepsilon = \epsilon^M$.
6. Note that the model M from the previous exercise contains an isomorphic copy of $\mathbb{R}$ (even as an L-structure). We will identify this copy with $\mathbb{R}$ itself.
7. Let $\langle\mathbb{R}\rangle \subseteq M$ be the convex hull of $\mathbb{R} \subseteq M$. Note that $x \in \langle\mathbb{R}\rangle$ if and only if $-n < x < n$ for some $n \in \mathbb{N}$. Show that for any such $x \in \langle\mathbb{R}\rangle \subseteq M$ there is a unique $r \in \mathbb{R}$ such that for all $m \in \mathbb{N}$ we have $-1/m < x - r < 1/m$. We denote it by $r = \mathrm{st}(x)$.
8. Prove that $\mathrm{st} : \langle\mathbb{R}\rangle \to \mathbb{R}$ has the following properties: $\mathrm{st}(x + y) = \mathrm{st}(x) + \mathrm{st}(y), \mathrm{st}(xy) = \mathrm{st}(x)\,\mathrm{st}(y)$. Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous function and $r \in \mathbb{R}$. Show that $\mathrm{st}(g^M(r + \varepsilon)) = g(r)$.
9. Let f be a function symbol corresponding to a differentiable function and let f' be the function symbol corresponding to its derivative. Show that for $r \in \mathbb{R}$ we have
$$\mathrm{st}\left(\frac{f^M(r+\varepsilon) - f^M(r)}{\varepsilon}\right) = f'(r).$$
In particular, show that the term inside st is in $\langle\mathbb{R}\rangle$.
10. Prove the Leibniz rule $(fg)' = f'g + fg'$ using the above interpretation of the derivative.
11. Read the exercises below. Familiarise yourself with their statements. Their solutions are optional.

Fix a countable first-order language L. The set of L-formulas (in countably many variables) is sometimes denoted by $L_{\omega,\omega}$. This is to indicate that when we form L-formulas, we can only consider finite conjunctions $\bigwedge_{i=1}^{n} \varphi_i$ (with $n < \omega$) and finite quantifications $\exists x_1 \dots \exists x_m \varphi$ (with $m < \omega$). Given infinite cardinals κ, λ with $\lambda \leq \kappa$ and a set V of variables with $|V| = \kappa$ one can consider $L_{\kappa,\lambda}$-preformulas which are defined similarly as $L_{\omega,\omega}$-formulas (so we start with atomic formulas and have negation) but with the following connectives and quantification steps:

- If $\{\varphi_\alpha\}_{\alpha\in A}$ are $L_{\kappa,\lambda}$-preformulas with $|A| < \kappa$ then so are
$$\bigwedge_{\alpha\in A} \varphi_\alpha, \bigvee_{\alpha\in A} \varphi_\alpha.$$
- If φ is a $L_{\kappa,\lambda}$-preformula and $X \subset V$ satisfies $|X| < \lambda$, then so is $\exists X\ \varphi$.

We distinguish among those $L_{\kappa,\lambda}$-formulas as those that have only fewer than λ free variables. In the following exercise we will be interested in $L_{\omega_1,\omega}$. In the proof system for $L_{\omega_1,\omega}$ we add axioms

$$\bigwedge_{\alpha\in A} \varphi_\alpha \to \varphi_{\alpha_0},$$

for any countable collection of formulas $\{\varphi_\alpha\}_{\alpha\in A}$ and $\alpha_0 \in A$, and the deduction rule scheme

$$\frac{\varphi_0, \varphi_1, \dots}{\bigwedge_{i\in\mathbb{N}} \varphi_i},$$

for every sequence of $L_{\omega_1,\omega}$-formulas $\{\varphi_i\}_{i\in\mathbb{N}}$. Here proofs are infinitely branching trees.

Exercise 1. ⓔ Imitate Henkin's proof to show that if an $L_{\omega_1,\omega}$-sentence σ is consistent, then it has a model. Conclude with the completeness theorem for infinitary formulas $\models \varphi \iff \vdash \varphi$.

Some systems of axioms use more than one type of primitive object. For example, Euclid's axioms are about points P, lines l, and the incidence relation between them $P \in l$. One can formalise this in the framework of multisorted (first-order) logic, where a language L consists of:

- Sort symbols $\{S_\alpha\}_{\alpha\in A}$.
- Function symbols $\{f_i\}_{i\in I}$, each equipped with its arity, which is a tuple $(\alpha_1, \dots, \alpha_n; \beta)$ for some $n \in \mathbb{N}$ and $\alpha_1, \dots, \alpha_n, \beta \in A$.
- Predicate symbols $\{P_j\}_{j\in J}$, each equipped with its arity which is a tuple $(\alpha_1, \dots, \alpha_m)$ for some $m \in \mathbb{N}$ and $\alpha_1, \dots, \alpha_m \in A$.
- Constant symbols $\{c_k\}_{k\in K}$, each equipped with a sort of index $\alpha = \alpha(k) \in A$.

In that case, an L-structure M is a collection of non-empty sets S_α^M together with interpretations of other symbols, where if f is a function symbol with arity $(\alpha_1, \dots, \alpha_n; \beta)$, then its interpretation is a function

$$f^M : S^M_{\alpha_1} \times \cdots \times S^M_{\alpha_n} \to S^M_\beta.$$

Similarly, if P is a predicate symbol with arity $(\alpha_1, \dots, \alpha_m)$, then its interpretation is a subset

$$P^M \subseteq S^M_{\alpha_1} \times \cdots \times S^M_{\alpha_m}.$$

If c is a constant of sort S, then M is equipped with its interpretation $c^M \in S^M$. When defining L-formulas, we need to consider variables decorated by sort symbols so each sort has a different set of variables associated to it.

Exercise 2. ⓔ Convince yourself that Henkin's proof of completeness works for multisorted structures.

Exercise 3. ⓔ Look at Hilbert's axioms of Euclidean geometry and define an appropriate multisorted first-order language.

Unless stated otherwise, all further results in the course hold for multisorted structures, however for the ease of notation we will be working with a single-sorted language.

Many axiom systems involve a complete metric and so the real numbers in their definition. For example the axioms of Banach spaces or Hilbert spaces are such. Continuous (first-order) logic is a variant of first-order logic, where one can consider some real-valued predicates and the equality is replaced by the metric. In the following by a *continuity modulus* we mean a continuous increasing function

$$\delta : [0,1] \to [0,1]$$

satisfying $\delta(0) = 0$ and $\delta(\varepsilon) > 0$ for $\varepsilon > 0$.

Definition 4. A *continuous language* L consists of:

- sort symbols S each equipped with a metric symbol d_S and with a bound $B_S \in \mathbb{R}_{\geq 0}$;
- constant symbols c with their arity as in the multisorted logic;
- function symbols f with the same arities as in the multisorted logic, each equipped with its continuity modulus δ_f;
- predicate symbols P with their arities same as in the multisorted logic, each equipped with its continuity modulus δ_P and with a bound $B_P \in \mathbb{R}_{\geq 0}$.

An L-structure M consists of interpretations of sorts, constant and function symbols, same as in multisorted logic, however, for each sort S its interpretation S^M is equipped with the metric symbol interpretation d_S^M which makes (S^M, d_S^M) a non-empty complete metric space of diameter at most B_S. Moreover, the function symbols interpretations are supposed to respect the continuity modulus in the following sense: for all real ε with $0 < \varepsilon \leq 1$, for every $a_1, \dots, a_n, b_1, \dots, b_n$ in appropriate sorts of M, we have

$$\max_{i=1,\dots,n} d(a_i, b_i) < \delta_f(\varepsilon) \implies d(f(a_1, \dots, a_n), f(b_1, \dots, b_n)) \leq \varepsilon,$$

where we skipped interpretation in M (and the indexes for sorts) in the notation. Furthermore, the interpretations of predicate symbols are functions

$$P^M : S^M_{\alpha_1} \times \dots \times S^M_{\alpha_m} \to [0, B_P]$$

that are required to respect the continuity modulus in the following sense: for all real $1 \geq \varepsilon > 0$ for every $a_1, \dots, a_m, b_1, \dots, b_m$ in appropriate sorts of M we have

$$\max_{i=1,\dots,m} d(a_i, b_i) < \delta_P(\varepsilon) \implies |P(a_1, \dots, a_m) - P(b_1, \dots, b_m)| \leq \varepsilon.$$

When we build formulas in continuous logic, terms are defined similarly as in the discrete case. More complicated expressions are constructed by:

(i) atomic formulas - if t_1, t_2 are terms of the same sort S, then $d_S(t_1, t_2)$ is an atomic formula. If $t_1, \dots, t_n$ are terms of appropriate sorts, and P is a predicate symbol whose arity has matching sorts, then $P(t_1, \dots, t_n)$ is an atomic formula.
(ii) connectives - if $\varphi_1, \dots, \varphi_n$ are formulas and $u : \mathbb{R}^n \to \mathbb{R}$ is a continuous function, then $u(\varphi_1, \dots, \varphi_n)$ is a formula.
(iii) quantifiers - if φ is a formula then so are $\sup_x \varphi$ and $\inf_x \varphi$.

In some variants of continuous logic one restricts only to $[0,1]$-valued formulas (all the bounds are equal to 1), and connectives are taken to be only $x \mapsto 1 - x, x \mapsto x/2, (x, y) \mapsto \max(x - y, 0)$. If M is an L-structure, $\varphi(\overline{x})$ is a formula with free-variables $\overline{x}$ and $\overline{a} \in M^{\overline{x}}$, then one inductively defines $\varphi(\overline{a})^M \in \mathbb{R}$. A sentence is said to hold in M if it is evaluated to zero. It is possibly to formalise the notion of a proof in continuous logic so that a completeness theorem holds, see [1]. Later, we may prove the following theorem, using ultraproducts.

Theorem 5 (Compactness for continuous logic)**.** Let T be a continuous logic theory. Assume that for every $\varphi_1, \ldots, \varphi_n \in T$ and every $\varepsilon > 0$ there is an L-structure M such that for all $i = 1, \ldots, n$ we have $|\varphi_i^M| \leq \varepsilon$. Then T has a model.

Exercise 6. e Axiomatize the following structures in continuous logic: Banach spaces, Hilbert spaces, C*-algebras. Hint: introduce a sort for each ball of radius $n \in \mathbb{N}$.