Exercises, sheet 2. Fix a first-order language L. M and N will denote L-structures and T will denote an L-theory. If $(I,<)$ is a linear order, by a (an elementary) chain of L-structures we mean a collection $(M_i)_{i\in I}$ such that $M_i \subseteq M_j$ if $i<j$ (resp. $M_i \prec M_j$).

1. Show that any for any finite structure in a finite language there is a sentence that determines it up to an isomorphism.
2. Show for an elementary chain $(M_i)_{i\in I}$, each M_i is elementarily embedded in the union of all M_j's (define a natural L-structure on this union). Generalise to the case where $(I,<)$ is a directed system.
3. For a theory T let $T_\forall$ be the set of universal sentences that are consequences of T. Show that $M \models T_\forall$ if and only if there is an $N \models T$ with $M \subseteq N$.
4. Deduce that a formula $\varphi(\overline{x})$ is T-equivalent to a universal formula if and only if for every $N \subseteq M$, $M \models T$ and $\overline{a} \in N^{|\overline{x}|}$, we have
$$M \models \varphi(\overline{a}) \implies N \models \varphi(\overline{a}).$$
5. Show that $\mathrm{Diag}_{\mathrm{el}}(M)$ (resp. $\mathrm{Diag}(M)$) is consistent with T, then there is an elementary embedding (resp. an embedding of L-structures) of M in a model of T.
6. We call T *model complete* if for every model $M \models T$, its atomic diagram is a complete theory. Show that the following are equivalent:
   - T is model complete,
   - every embedding of models of T is elementary,
   - every formula is T-equivalent to a universal one,
   - every formula is T-equivalent to an existential one.
7. Show that a theory T can be axiomatized by $\forall\exists$-formulas (i.e., T is *inductive*) if and only if for any chain of L-structures $(M_i)_{i\in I}$ that are models of T, their union is also a model of T.
8. A model $M \models T$ is called *existentially closed* if for any extension $M \subseteq N$ with $N \models T$, M is *existentially closed in* N (written $M \prec_\exists N$), i.e., whenever $\varphi(\overline{y})$ is an existential formula and $\overline{b} \in M^{|\overline{y}|}$, then
$$M \models \varphi(\overline{b}) \iff N \models \varphi(\overline{b}).$$
   Show that if T is inductive, then any model of T can be embedded in an existentially closed model of T. Show that T is model complete if and only if all of its models are existentially closed.
9. An L-theory $T^\star$ is a *model companion* of T if:
   - every model of T admits an embedding into a model of $T^\star$,
   - every model of $T^\star$ admits an embedding into a model of T,
   - $T^\star$ is model complete.

   Show that if a model companion of T exists, it is unique up to equivalence of theories.
10. Show that if T is inductive, its model companion exists if and only if the class of its existentially closed models is axiomatized by a theory (i.e., this class is *elementary*).