

Exercises, sheet 8. Let T be a complete theory in a countable language L , with a monster model $\mathfrak{C}$. Unless stated otherwise, all parameter sets and sequences are small. Definable sets may use parameters from $\mathfrak{C}$. In Exercises 3,4 assume that T is ω -stable, and in Exercises 6,7,8 assume that T is stable.

1. Show that $\text{MR}(X) = 0$ if and only if X is non-empty and finite, and that in this case $\text{MD}(X) = |X|$.
2. Let X be a definable set of cardinality m , where $1 \leq m < \omega$. Prove that

$$\text{TR}(X) = \lfloor \log_2 m \rfloor.$$

Show that if X is strongly minimal, then $\text{TR}(X) = \omega$.

3. Prove the following properties of the two-rank.
 - (i) If $p(x)$ is a consistent partial type over $A \subseteq B$, there is at most one $q \in S_x(B)$ such that $p \subseteq q$ and $\text{TR}(q) = \text{TR}(p)$.
 - (ii) For every $p \in S_x(A)$ there is a finite subset $A_0 \subseteq A$ such that

$$\text{TR}(p|_{A_0}) = \text{TR}(p).$$

Deduce that p is the unique extension of $p|_{A_0}$ to a complete type over A with the same two-rank.

- (iii) Let $\varphi(x, a)$ be a consistent formula over A . Show that a type of minimal two-rank in $[\varphi(x, a)] \subseteq S_x(A)$ is isolated. Conclude that isolated types are dense in $S_x(A)$.
4. Let T be the theory of an equivalence relation E with exactly two classes, both infinite, and let $a \in \mathfrak{C}$. Show that there is a unique type $p \in S_1(\emptyset)$, and find all its extensions to complete types over $\{a\}$. Compute the Morley rank and the two-rank of p and of each extension. Deduce that preserving Morley rank does not imply uniqueness of an extension, and that an extension preserving two-rank need not exist.
5. Solve Exercise 11.12.
6. Let $I \subseteq \mathfrak{C}$ be an infinite indiscernible set over A , and let $B \supseteq A$. Define

$$\text{Av}(I/B) = \{\varphi(x, b) \in L(B) : \mathfrak{C} \models \varphi(a, b) \text{ for all but finitely many } a \in I\}.$$

- (i) Use Theorem 12.2 to show that $\text{Av}(I/B)$ is a complete type over B . Show that it is finitely satisfiable in I , meaning that every finite subset is realised by an element of I .
 - (ii) Show that for every infinite subset $J \subseteq I$,

$$\text{Av}(J/B) = \text{Av}(I/B).$$

- (iii) If c realises $\text{Av}(I/A \cup I)$, show that $c \notin I$ and that $I \cup \{c\}$ is indiscernible over A . The type $\text{Av}(I/B)$ is called the *average type* of I over B .

7. Let I be as in the previous exercise, and put $p = \text{Av}(I/\mathfrak{C})$. Fix an L -formula $\varphi(x, y)$ and let $r = r_\varphi$ be given by Theorem 12.2. For any distinct $a_1, \dots, a_{2r+1} \in I$, find an explicit formula $\psi(y, a_1, \dots, a_{2r+1})$ such that for every $b \in \mathfrak{C}^y$,

$$\varphi(x, b) \in p \iff \mathfrak{C} \models \psi(b, a_1, \dots, a_{2r+1}).$$

Hint: use a majority vote.

Deduce that for every fixed countably infinite $J \subseteq I$, the type p is *definable over J* : for every $\varphi(x, y)$, the set $\{b : \varphi(x, b) \in p\}$ is definable with parameters from J .

8. Let I and B be as in Exercise 6. Show that

$$|\{a \in I : a \text{ does not realise } \text{Av}(I/B)\}| \leq |B| + \aleph_0.$$

Deduce that if $|I| > |B| + \aleph_0$, then I contains $|I|$ realisations of $\text{Av}(I/B)$. Show that the strict inequality cannot be dropped, even for the theory of infinite sets in the language consisting only of equality.