**Exercises, sheet 0, o-minimality.** Fix a field $(F,+,\cdot)$. A subset $P \subseteq F$ is called a *cone* if $P+P \subseteq P, F^2 \subseteq P$. It is proper if $-1 \notin P$. We order cones in $F$ by inclusion. An *order* on $F$ is a linear order $x \leqslant y$ that is preserved by adding the same element on both sides, and such that the product of two non-negative elements is non-negative. Show that:

1. If $\leqslant$ is an ordering on $F$, then $P_0 = \{x \in F : x \geqslant 0\}$ is a maximal cone. Moreover $P_0 \cup -P_0 = F$.
2. If $P$ is proper and $P \cup -P = F$, then $x \leqslant y \iff y - x \in P$ defines an order on $F$.
3. Every proper cone is contained in a maximal one. Deduce that a proper cone $P$ is maximal if and only if $F = P \cup -P$.
4. The field $F$ has an order if and only if $-1 \notin \Sigma F^2$ ($-1$ is not a sum of squares). Such fields are called *formally real*.
5. If $F$ is formally real, $a \in F^\times$ and $F[\sqrt{a}]$ is not formally real, then $-a \in \Sigma F^2$. Deduce that for every $a$ as above, either $F[\sqrt{a}]$ or $F[\sqrt{-a}]$ is formally real.
6. A field is called *real closed* if it does not have a proper algebraic formally real extension. Show that in a real closed field $P = \{x^2 \in F : x \in F\}$ is a maximal proper cone, so it defines (a unique) order.
7. Assume that $F$ is real closed and let $f(x) \in F[x]$ be an odd degree polynomial. Show that $f$ has a root in $F$. Hint: induction on the degree.
8. If $F$ is real closed, then $F[\sqrt{-1}]$ is algebraically closed. Use Galois theory or induction. Conclude that every polynomial over $F$ is a product of linear and quadric terms.
9. If $F[\sqrt{-1}]$ is algebraically closed, then $F$ is real closed. Hence we have shown that $F[\sqrt{-1}]$ is algebraically closed if and only if $F$ is real closed.
10. Assume that $F$ is real closed. Let $f \in F[x]$ and assume that $a < b, f(a) < c < f(b)$ with respect to the unique order on $F$. Show that there is $d \in F$ with $a < d < b$ and $f(d) = c$.
11. Write first-order axioms for real closed fields.