**Exercises, sheet 0, valued fields.** Fix a field $(F, +, \cdot)$. An absolute value $|\cdot| : F \to \mathbb{R}_{\geq 0}$ is a function that satisfies:

- $|x| = 0 \iff x = 0$,
- $|xy| = |x| \cdot |y|$,
- $|x + y| \leq |x| + |y|$.

It is called non-Archimedean if additionally $|x + y| \leq \max(|x|, |y|)$. On $\mathbb{Q}$ we define $|q|_p = p^{-k}$, if one can write $q = p^k \cdot \frac{a}{b}$ with $(p, a) = (p, b) = 1$. For a field $D$, we define $|\cdot|_t : D(t) \to \mathbb{R}_{\geq 0}$, by requiring that $\left|\sum_{i=k}^{n} a_i t^i\right|_t = \exp(-k)$ if $a_k \neq 0$, and by requiring multiplicativity (with $|0|_t = 0$). We assume that $(F, |\cdot|)$ is a field equipped with an absolute value.

1. Verify that $|\cdot|_p$ is a non-Archimedean absolute value on $\mathbb{Q}$ and $|\cdot|_t$ is an absolute value on $D(t)$.
2. Show that if $|\cdot|$ is a non-Archimedean absolute value on $\mathbb{Q}$, then $|\cdot| = |\cdot|_p^r$ for some prime $p$ and some $r \in \mathbb{R}_{\geq 0}$ (with the convention $0^0 = 0$). *Show that every Archimedean absolute value on $\mathbb{Q}$ is of the form $|\cdot|_\infty^r$ for some $r \in (0, 1]$. Together this yields Ostrowski's theorem.
3. Show that $d(x, y) = |x - y|$ defines a metric on $F$. Note that this metric is an ultrametric if and only if $|\cdot|$ is non-Archimedean.
4. Show that $|\cdot|$ is non-Archimedean if and only if for every $n \in \mathbb{Z}$, $|n| \leq 1$. Observe that $\mathcal{O} = \{x \in F : |x| \leq 1\}$ forms a subring of $F$, such that for every $x \in F^\times$ either $x \in \mathcal{O}$ or $x^{-1} \in \mathcal{O}$. Domains with this property holding inside their fraction field are called *valuation rings*. If a valuation ring is contained in a field and it generates it, we say that it is a *valuation subring* of that field. Write first-order axioms for fields together with a valuation subring.
5. Let $\widehat{F}$ be the set of equivalence classes of Cauchy sequences, with respect to the limit of distances being zero. Equip it with the field structure and with an absolute value. Show that the corresponding metric space of $\widehat{F}$ is complete. It is called the completion of $(F, |\cdot|)$.
6. Let $\mathcal{O} = \{x \in F : |x| \leq 1\} \subseteq F$ and $\widehat{\mathcal{O}} = \{x \in \widehat{F} : |x| \leq 1\} \subseteq \widehat{F}$. Assume that there is $0 \neq \varpi \in F$ such that $|\varpi| < 1$. Show that as a ring $\widehat{\mathcal{O}}$ is isomorphic to $\varprojlim_n \mathcal{O}/(\varpi^n)$. Furthermore, show that $\mathcal{O}/(\varpi^n) \simeq \widehat{\mathcal{O}}/(\varpi^n)$ and $F = \mathcal{O}[\varpi^{-1}]$, $\widehat{F} = \widehat{\mathcal{O}}[\varpi^{-1}]$.
7. Let $\mathbb{Q}_p$ be the completion of $\mathbb{Q}$ with respect to $|\cdot|_p$. Identify the valuation ring of $\mathbb{Q}_p$ with the ring $\mathbb{Z}_p = \varprojlim_n \mathbb{Z}/(p^n)$. Show that the metric topology coincides with the inverse limit topology (of discrete finite sets). What is the completion of $D(t)$ with respect to $|\cdot|_t$?
8. Let $K$ be a field and let $\mathcal{O} \subseteq K$ be a valuation subring (not necessarily coming from an absolute value). Recall that an abelian group order is an order on an abelian group that is preserved by adding the same element on both sides. Show that:
   - every finitely generated ideal in $\mathcal{O} \subseteq K$ is principal;
   - the abelian group $\Gamma = K^\times/\mathcal{O}^\times$ is totally ordered by $a/\mathcal{O}^\times \leqslant b/\mathcal{O}^\times \iff ba^{-1} \in \mathcal{O}^\times$;
   - the tautological function $v : K \to \Gamma \cup \{\infty\}$ (that only send zero to infinity) satisfies
   $$v(xy) = v(x) + v(y), v(x + y) \geqslant \min(v(x), v(y));$$
   - if $\iota : \Gamma \to \mathbb{R}$ is an embedding of ordered abelian groups, then for $|x| = \exp(-\iota(v(x)))$, $|\cdot|$ is an absolute value on $K$.
9. Show that if $(\Gamma, + \leqslant)$ is a totally ordered abelian group and $v : K \to \Gamma \cup \{\infty\}$ (that only sends zero to infinity) satisfies $v(xy) = v(x)+v(y), v(x+y) \geqslant \min(v(x), v(y))$, then $\mathcal{O} := \{x \in K : v(x) \geqslant 0\}$ is a valuation subring of $K$. Furthermore, show that $v$ factors through the tautological map $K \to K^\times/\mathcal{O}^\times \cup \{\infty\}$. A function $v$ with properties described above is called a *valuation* on $K$.
10. Show that if the absolute value $|\cdot|$ is non-Archimedean, then $-\log|\cdot| : F \to \mathbb{R} \cup \{\infty\}$ is a valuation on $F$.
11. A *place* on a field $K$ is a function $f : K \to \kappa \cup \{\infty\}$, where $\kappa$ is a field, such that $f(xy) = f(x)f(y), f(x) + f(x) = f(x + y), f(1) = 1, f(0) = 0$ where $\infty \cdot 0, 0 \cdot \infty$ can be anything, and $a + \infty = \infty + a = \infty$ for all $a \in \kappa$. Find a bijective correspondence between places of $K$ and valuation subrings of $K$.